Further Topics in Statistics and Probabilities

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This class

In this class, we will review the fundamental concepts of probability theory and apply them to problems in mathematical economics. We will notably discuss models of interactive knowledge and belief revision as they have been introduced in economics by Robert Aumann's article "Agreeing to disagree" (1976).

Evaluation is based on:

- your participation during the sessions,
- written assignments and presentations in class, and
- a final exam.

Motivation

As individuals participating in society (consumers, investors, voters, citizens) we are frequently faced with situation in which we have to attribute probabilities to certain events without being able to derive those probabilities from a clearly defined underlying mathematical model, such as the throw of a dice.

Instead, we have to exploit subjective information that we acquire about the state of the world.

Still we are rational. We want to come up with these subjective probabilities not in an arbitrary way. Rather we want to exploit in a coherent and rational way all information available to us.

The Bayesian approach to probabilities offers a model for that.

What complicates the picture is that we interact with others: When I observe you acting in a certain way, I might deduce from that information about the state of the world, which allows me to update the probabilities that I attribute to certain events.

This is the topic of this class: We will consider individuals who update their beliefs about certain events in a Bayesian rational way—using Bayes' Law—by exploiting the information that is available to them and that they deduce from observing the actions of other individuals.

In doing so, we will make use of some of the basic concepts of probability theory (which you have seen last year in your class Statistics 2 and which you are currently seeing in Statistics 3).

Chapter 2

The formal framework and Aumann's theorem

The formal framework (Aumann 1976)

Let $(\Omega, \mathscr{B}, p)$ be a probability space:

- Ω the set of possible states of the world (a generic element of which is often denoted by $\omega \in \Omega$),
- $\mathscr{B}$ a σ-algebra on Ω, and
- p the prior probability distribution defined on $(\Omega, \mathscr{B})$.

We consider two individuals, 1 and 2, who impute the same prior probability p to the events in $\mathscr{B}$ but who have access to private information, given by a finite partition $\mathscr{P}_i$ of Ω, $i \in \{1, 2\}$, that is, a finite set

$$\mathscr{P}_i = \{P_{i1}, P_{i2}, \ldots, P_{ik}, \ldots, P_{iK_i}\}$$

of nonempty subsets of Ω, the *classes* (or *cells*) of the partition, such that:

(a) each pair $(P_{ik}, P_{ik'})$, $k \neq k'$, is disjoint, that is, $P_{ik} \cap P_{ik'} = \emptyset$, and

(b) $\bigcup_k P_{ik} = \Omega$.

The partition $\mathscr{P}_i$ models individual i's information: when $\omega \in \Omega$ is the true state, the individual characterized by $\mathscr{P}_i$ will learn that one of the states that belong to the class of the partition $\mathscr{P}_i$ to which belongs ω, denoted by $P_i(\omega)$, has materialized.

In order to guarantee that the classes P_{ik} of the partition $\mathscr{P}_i$ are measured by p, we suppose, of course, that they belong to the σ-algebra $\mathscr{B}$ defined on Ω.

Example

Let $\Omega = \{a, b, c, d, e, f, g, h, i, j, k\}$ and

$$\mathscr{P}_i = \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}\}.$$

Assume $\omega^\star = c$, the true state of the world. Then individual i, modeled by the partition above, will only receive the information that the true state of the world is in $\{c, d, i, j\}$, that is, that one of the states in $\{c, d, i, j\}$ has materialized (but not which one exactly). In our notation: $P_i(c) = \{c, d, i, j\}$.

Methodological reflection: We, in the role of the theorist who builds the model, know that the true state is $\omega^\star = c$ (hence, we can write $P_i(c)$). But the individual in the model only knows that the true state is one of the states in $\{c, d, i, j\}$.

With this interpretation: if ω is the true state and $P_i(\omega) \subset A$, that is, $P_i(\omega)$ *implies* A, then individual i (at state ω) "knows" that event A has happened.

In the example above:

$$\mathscr{P}_i = \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}\}.$$

When $\omega^\star = c$ is the state that has materialized, i will *know* that the state that has materialized is in $\{c, d, i, j\}$, that is, that the event $\{c, d, i, j\}$ has occurred.

As a consequence, i will also know that any event that is a superset of $\{c, d, i, j\}$ has occurred. For example, i will also know that the event $\{a, c, d, i, j, e\}$ has occurred. And certainly, i will also know that any event disjoint of $\{c, d, i, j\}$ did not occur. For example, i will also know that the event $\{a, e\}$ *did not* occur.

A crucial assumption:

Following Aumann (1976), we assume that the prior p defined on $(\Omega, \mathscr{B})$ as well as the information partitions of the two individuals, $\mathscr{P}_i$, $i \in I = \{1, 2\}$, are *common knowledge* between the two individuals.

According to David Lewis (1969), an event is *common knowledge* between two individuals if not only both know it but also both know that the other knows it and that both know that the other knows that they both know it, ad infinitum (Lewis 1969).

A probabilistic model of "beliefs"

More generally, if individual i is Bayesian rational, then for any event A that belongs to the σ-algebra defined on Ω, after realization of the true state of the world, i can calculate the posterior probability of A given the information provided by the partition $\mathscr{P}_i$, that is, the *conditional probability of A given that the true state belongs to $P_i(\omega)$*:

$$q_i = p(A \mid P_i(\omega)) = \frac{p(A \cap P_i(\omega))}{p(P_i(\omega))}.$$

Remember: $P_i(\omega)$ denotes i's *information class* (or *cell* or *set*) to which belongs ω.

Example Let $\Omega = \{a, b, c, d, e, f, g, h, i, j, k, l, m\}$, endowed with uniform prior, that is, $p(\omega) = 1/13$ for all possible states, and

$$\begin{aligned} \mathscr{P}_1 &= \{\{a, b, c, d, e, f\}, \{g, h, i, j, k\}, \{l\}, \{m\}\}, \\ \mathscr{P}_2 &= \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}, \{l, m\}\}. \end{aligned}$$

Let $A = \{a, b, i, j, k\}$ be the event of interest, and suppose $\omega^\star = a$. Then:

$$\begin{aligned} q_1 = P(A \mid P_1(a)) &= \frac{p(\{a, b, i, j, k\} \cap \{a, b, c, d, e, f\})}{p(\{a, b, c, d, e, f\})} \\ &= \frac{p(\{a, b\})}{p(\{a, b, c, d, e, f\})} = \frac{\frac{2}{13}}{\frac{6}{13}} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} q_2 = P(A \mid P_2(a)) &= \frac{p(\{a, b, i, j, k\} \cap \{a, b, g, h\})}{p(\{a, b, g, h\})} \\ &= \frac{p(\{a, b\})}{p(\{a, b, g, h\})} = \frac{\frac{2}{13}}{\frac{4}{13}} = \frac{1}{2} \end{aligned}$$

Terminology:

In game theory, decision theory, and economics, the probability attributed to an event is also called a *belief*.

In this terminology, $p(A)$ is the *prior belief of* A, which by assumption is common knowledge between the two individuals, and $p(A \mid P_i(\omega))$ the *posterior belief* that i attributes to A given the information received through his or her partition.

Remember: According to David Lewis (1969), an event is *common knowledge* between two individuals if not only both know it but also both know that the other knows it and that both know that the other knows that they both know it, ad infinitum (Lewis 1969).

To capture this notion within a set-theoretic framework that relies on the notion of a state of the world, it turns out to be useful—and having established this is one of the main achievements of Aumann—to consider the *meet* of the two partitions.

Definition 1 Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two partitions of Ω. The *meet* of $\mathscr{P}_1$ and $\mathscr{P}_2$, denoted by $\hat{\mathscr{P}} = \mathscr{P}_1 \wedge \mathscr{P}_2$, is the *finest common coarsening* of $\mathscr{P}_1$ and $\mathscr{P}_2$, that is, the finest partition of Ω such that, for each $\omega \in \Omega$,

$$P_i(\omega) \subset \hat{P}(\omega), \quad \forall i \in I = \{1, 2\},$$

where $\hat{P}(\omega) = P_1 \wedge P_2(\omega)$ is the class of the meet to which belongs ω.

Example

$$\begin{aligned} \mathscr{P}_1 &= \{\{a, b, c, d, e, f\}, \{g, h, i, j, k\}, \{l\}, \{m\}\}, \\ \mathscr{P}_2 &= \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}, \{l, m\}\}. \end{aligned}$$

$\hat{\mathscr{P}} = \mathscr{P}_1 \wedge \mathscr{P}_2 = \{\{a, b, c, d, e, f, g, h, i, j, k\}, \{l, m\}\}$

The meet of the two information partitions, casually speaking, represents what is *common knowledge* between the two individuals. The following Lemma makes this more precise.

Lemma 1 (Aumann 1976) An event $A \subset \Omega$, at state ω, is *common knowledge* between individuals 1 and 2 in the sense of the recursive definition (Lewis 1969) if and only if $\hat{P}(\omega) \subset A$, that is: if the information class of the meet of the two partitions to which belongs ω is contained in A.

Definition 2 Let $\mathscr{P}_1$ and $\mathscr{P}_2$ partitions of Ω. We say that $\omega' \in \Omega$ *can be reached* from another element $\omega \in \Omega$ if there is a sequence of subsets of Ω, $P^1, P^2, \ldots, P^n, \ldots, P^N$ such that $\omega \in P^1$ and $\omega' \in P^N$ and consecutive P^n, in an alternating manner, belong to $\mathscr{P}_1$ and $\mathscr{P}_2$ and intersect with each other.

Démonstration (Lemma 1, Aumann 1976): At ω, Individual 1 knows that A happend if $P_1(\omega) \subset A$. Suppose that this is the case and let $P^1 = P_1(\omega)$. Individual 1 knows that individual 2 knows that A happend if all P_{2k} with nonempty intersection with P^1 are a subset of A. Two cases: (1) If for all P_{2k} with nonempty intersection with P^1, the intersection with P^1 is P_{2k} itself, then P^1 contains all $\omega' \in \Omega$ that can be reached from ω: P^1 will be an element of the meet $\mathscr{P}_1 \wedge \mathscr{P}_2$, and all sentences of the form "i knows that j knows that i knows ... A" are true, that is, A will be common knowledge. (2) If not, let P^2 be a P_{2k} whose nonempty intersection with P^1 is not P_{2k} itself. Individual 1 knows that individual 2 knows that individual 1 knows A if all P_{1k} with nonempty intersection with P^2 are subsets of A.

Again two cases: (1) If for all P_{1k}, different from P^1, the intersection with P^2 is P_{1k} itself, then $P^1 \cup P^2$ contains all ω' that can be reached from ω: $P^1 \cup P^2$ will be an element of the meet $\mathscr{P}_1 \wedge \mathscr{P}_2$, and all sentences of the form "*i* knows that *j* knows that *i* knows ... *A*" are true, that is, *A* will be common knowledge. (2) If not, let P^3 be a $P_{1k} \in \mathscr{P}_1$, different from P^1, for which the intersection with P^2 is not P_{1k} itself, and so on. We see that all sentences of the form "*i* knows that *j* knows that *i* knows ... *A*" are true if and only if *A* contains all ω' that can be reached from ω. But the set of all ω' that can be reached from ω is an element of the meet $\mathscr{P}_1 \wedge \mathscr{P}_2$. QED.

Remark. Of course, if P is a class of the meet $\mathscr{P}_1 \wedge \mathscr{P}_2$, then, the union of all classes P_{ik} of the partition $\mathscr{P}_i$ contained in P is P,

$$\bigcup_{P_{ik} \subset P} P_{ik} = P,$$

and hence $\mathscr{P}_i$ induces a partition of P.

This is easy to verify in the example from above:

$$\begin{aligned} \mathscr{P}_1 &= \{\{a,b,c,d,e,f\},\{g,h,i,j,k\},\{l\},\{m\}\}, \\ \mathscr{P}_2 &= \{\{a,b,g,h\},\{c,d,i,j\},\{e,f,k\},\{l,m\}\}. \end{aligned}$$

$\hat{\mathscr{P}} = \mathscr{P}_1 \wedge \mathscr{P}_2 = \{\{a,b,c,d,e,f,g,h,i,j,k\},\{l,m\}\}$

Posteriors as "events"

Example

$\Omega = \{a, b, c, d, e, f, g, h, i, j, k, l, m\}$, with uniform prior,

$$\begin{aligned}
\mathscr{P}_1 &= \{ \overbrace{\{a,b,c,d,e,f\}}^{p(A|\{a,b,c,d,e,f\})=1/3}, \overbrace{\{g,h,i,j,k\}}^{p(A|\{g,h,i,j,k\})=3/5}, \overbrace{\{l\}}^{p(A|\{l\})=0}, \overbrace{\{m\}}^{p(A|\{m\})=0} \},\\
\mathscr{P}_2 &= \{ \underbrace{\{a,b,g,h\}}_{p(A|\{a,b,g,h\})=1/2}, \underbrace{\{c,d,i,j\}}_{p(A|\{c,d,i,j\})=1/2}, \underbrace{\{e,f,k\}}_{p(A|\{e,f,k\})=1/3}, \underbrace{\{l,m\}}_{p(A|\{l,m\})=0} \},
\end{aligned}$$

Let $A = \{a, b, i, j, k\}$. For individual 1: attributing to A a posterior of $1/3$ corresponds to the event $\{a, b, c, d, e, f, \}$; attributing to A a posterior of 0 corresponds to the event $\{l, m\}$; attributing to A a nonzero posterior corresponds to the event $\{a, b, c, d, e, f, g, h, i, j, k\}$

For individual 2, attributing to A a posterior of $1/2$ corresponds to the event $\{a, b, c, d, g, h, i, j\}$, etc.

Common knowledge of posteriors

Suppose that $\omega^\star = m$ the true state of the world. Then, individual 1 will attribute to A a posterior of 0. This fact will be common knowledge between the two individuals, even though individual 2 does not know whether 1 has received the information that the true states belongs to $\{l\}$ or to $\{m\}$. This will be so, because for any of these two cases, individual 1 will always have calculated a posterior of 0.

At the same time, individual 2 will attribute to A a posterior of 0, and this will also be common knowledge.

Aumann's (1976) "agreement" result

Robert Aumann, (1976) "Agreeing to disagree," *The Annals of Statistics* 4 (6): 1236-1239.

- In economics, Aumann's paper has stimulated a rich literature.
- Derives its importance also for the formal framework that it proposes for modeling knowledge and common knowledge (the model relying on information partitions that we discuss in this class).
- What is this result?

Proposition (Aumann 1976)

Let $(\Omega, \mathscr{B}, p)$ a probability space, $\mathscr{P}_1$ and $\mathscr{P}_2$ two finite partitions of Ω, measurable with respect to $\mathscr{B}$, that represent the information accessible to individual 1 respectively 2, all of this being common knowledge between the two individuals. Let furthermore $A \in \mathscr{B}$ be an event. If at state ω (in virtue of the common knowledge of the prior probability and the information partitions) the posteriors q_1 and q_2 that the individuals attribute to A are common knowledge, then they have to be equal: that is, $q_1 = q_2$.

The proof

Can be understood in three steps. Step 1—conceptually the most important—consists in establishing that common knowledge of q_i implies that for any information class of $\mathscr{P}_i$ that is a subset of the information class of the meet to which belongs the true state, $P_i(\omega)$, the conditional probability of A has to be equal to q_i:

$$q_i = \frac{p(A \cap P_i(\omega))}{p(P_i(\omega))} = \frac{p(A \cap P_{ik})}{p(P_{ik})}, \quad \forall\, P_{ik} \subset \hat{P}(\omega). \tag{1}$$

Otherwise there would be some level of knowledge at which q_i would not be known, and therefore cannot be common knowledge.

Illustration:

$$\begin{aligned}
\mathscr{P}_1 &= \{ \overbrace{\{a, b\}}^{p(A|\{a,b\})=\frac{1}{2}}, \overbrace{\{c, d\}}^{p(A|\{c,d\})=\frac{1}{2}}, \{e\}, \{f\}\} \\
\mathscr{P}_2 &= \{ \{a, c\}, \{b, d\}, \{e, f\}\},
\end{aligned}$$

where $A = \{b, c\}$, and a the true state; $\hat{P}(a) = \{a, b, c, d\}$

Step 2: From (1) and the fact that the classes of i's partition that are subsets $\hat{P}(\omega)$ induce a partition of $\hat{P}(\omega)$, one obtains that:

$$q_i = \frac{p(A \cap \hat{P}(\omega))}{p(\hat{P}(\omega))}. \tag{2}$$

To see why (2) holds, note that (1) can be written as

$$p(A \cap P_{ik}) \quad = \quad q_i \, p(P_{ik}), \qquad \forall \, P_{ik} \subset \hat{P}(\omega).$$

Summing over all $P_{ik} \subset \hat{P}(\omega)$ gives

$$\sum_{P_{ik} \subset \hat{P}(\omega)} p(A \cap P_{ik}) \quad = \quad q_i \sum_{P_{ik} \subset \hat{P}(\omega)} p(P_{ik}).$$

Since the P_{ik} are disjoint (because they are elements of a partition), and the union over all P_{ik}-s that are subsets of $\hat{P}(\omega)$ gives $\hat{P}(\omega)$, by the property of σ-additivity of the probability measure p we have:

$$p(A \cap \hat{P}(\omega)) \quad = \quad q_i \, p(\hat{P}(\omega)).$$

Rearranging terms gives equation (2).

Step 2 relies on the more general fact that if A_k is a sequence of disjoint subsets of Ω and $p(B \mid A_k) = q$ for all k, then $p(B \mid \cup A_k) = q$, which is a simple consequence of the Kolmogorov Axioms.

Illustration:

$$\begin{aligned}
\mathscr{P}_1 &= \{ \overbrace{\overbrace{\{a,\ b\}}^{p(A|\{a,b\})=\frac{1}{2}},\ \overbrace{\{c,\ d\}}^{p(A|\{c,d\})=\frac{1}{2}}}^{p(A|\{a,b,c,d\})=\frac{1}{2}}, \{e\}, \{f\}\} \\
\mathscr{P}_2 &= \{\{a,\ c\}, \{b,\ d\}, \{e, f\}\}
\end{aligned}$$

Step 3: Finally, from the fact that (2) has to hold for each of the two individuals, one obtains that:

$$q_1 = \frac{p(A \cap \hat{P}(\omega))}{p(\hat{P}(\omega))} = q_2. \tag{3}$$

which concludes the proof.

Illustration:

$$\begin{array}{rcl}
\mathscr{P}_1 & = & \{ \overbrace{\overbrace{\{ a, b \}}^{p(A|\{a,b\})=\frac{1}{2}}, \overbrace{\{ c, d \}}^{p(A|\{c,d\})=\frac{1}{2}}}^{p(A|\{a,b,c,d\})=\frac{1}{2}}, \{e\}, \{f\}\} \\
\mathscr{P}_2 & = & \{ \underbrace{\underbrace{\{ a, c \}}_{p(A|\{a,c\})=\frac{1}{2}}, \underbrace{\{ b, d \}}_{p(A|\{b,d\})=\frac{1}{2}}}_{p(A|\{a,b,c,d\})=\frac{1}{2}}, \{e, f\}\}
\end{array}$$

The Aumann conditions

Putting (1)–(3) together, one has:

$$q_i = \frac{p(A \cap P_i(\omega))}{p(P_i(\omega))} = \frac{p(A \cap P_{ik})}{p(P_{ik})} = \frac{p(A \cap \hat{P}(\omega))}{p(\hat{P}(\omega))} \quad \forall P_{ik} \subset \hat{P}(\omega), \quad \forall i \in$$

That is, for each i, the posterior attributed to A, given $P(\omega)$, has to be equal to:

(1) the posterior probability of A given any of the classes P_{ik} of i's partition that are contained in the class of the meet to which belongs the true state of the world $\hat{P}(\omega)$, and
(2) the posterior probability of A given $\hat{P}(\omega)$, that is, the element of the meet to which belongs ω.

I refer to equation (4) as the *Aumann conditions*.

Example (in which the Aumann conditions hold)

$$\begin{aligned} \mathscr{P}_1 &= \{\{a,b\},\{c,d\},\{e\},\{f\}\}, \\ \mathscr{P}_2 &= \{\{a,c\},\{b,d\},\{e,f\}\}, \end{aligned}$$

$A = \{b, c\}$ the event of interest, and $\omega = a$ the true state of the world. Uniform prior, that is, $1/6$ for each possible state of the word. Then:

$$\begin{aligned} q_1 &= \frac{p(A \cap P_1(a))}{p(P_1(a))} = \frac{p(\{b,c\} \cap \{a,b\})}{p(\{a,b\})} = \frac{p(\{b\})}{p(\{a,b\})} = \frac{1}{2} \\ q_2 &= \frac{p(A \cap P_2(a))}{p(P_2(a))} = \frac{p(\{b,c\} \cap \{c,a\})}{p(\{c,a\})} = \frac{p(\{c\})}{p(\{c,a\})} = \frac{1}{2} \end{aligned}$$

The meet is $\hat{\mathscr{P}} = \{\{a, b, c, d\}, \{e, f\}\}$. Hence, $\hat{P}(a) = \{a, b, c, d\}$. Here, each i thinks it possible that the other has received any of the classes in the others partition that are included in $\hat{\mathscr{P}} = \{\{a, b, c, d\}$. However:

$$\begin{aligned}
\frac{p(\{b,c\} \cap \{c,d\})}{p(\{c,d\})} &= \frac{p(\{c\})}{p(\{c,d\})} = \frac{1}{2}, \\
\frac{p(\{b,c\} \cap \{d,b\})}{p(\{d,b\})} &= \frac{p(\{b\})}{p(\{d,b\})} = \frac{1}{2}.
\end{aligned}$$

And, as it should be according to the Aumann conditions:

$$p(\{b,c\} \mid \hat{P}(a)) = \frac{p(\{b,c\} \cap \{a,b,c,d\})}{p(\{a,b,c,d\})} = \frac{p(\{b,c\})}{p(\{a,b,c,d\})} = \frac{1}{2}.$$

Illustration:

$$
\begin{aligned}
\mathscr{P}_1 &= \{ \overbrace{\overbrace{\{a, b\}}^{p(A|\{a,b\})=\frac{1}{2}}, \overbrace{\{c, d\}}^{p(A|\{c,d\})=\frac{1}{2}}}^{p(A|\{a,b,c,d\})=\frac{1}{2}}, \{e\}, \{f\}\} \\
\mathscr{P}_2 &= \{ \underbrace{\underbrace{\{a, c\}}_{p(A|\{a,c\})=\frac{1}{2}}, \underbrace{\{b, d\}}_{p(A|\{b,d\})=\frac{1}{2}}}_{p(A|\{a,b,c,d\})=\frac{1}{2}}, \{e, f\}\}
\end{aligned}
$$

Chapter 3

Direct communication

Imagine that after realisation of the true state of the world the two individuals communicate to each other the information class of his or her partition of which they have learnd that the true state of the world belongs to it. Such an exchange of information can be referred to as one of direct communication (see, for instance, Geanakoplos and Polemarchakis 1982).

What the individuals know after such an exchange is given by the intersection of the two respective classes of their information partitions. Over the entire range of Ω, the so defined set of subsets of Ω is given by the coarsest common refinement of the two partitions: their so-called *join*.

Definition 2 Let $\mathscr{P}_1$ and $\mathscr{P}_2$ two partitions of Ω. The *join* of $\mathscr{P}_1$ and $\mathscr{P}_2$, denoted by $\check{\mathscr{P}} = \mathscr{P}_1 \vee \mathscr{P}_2$, is the *coarsest common refinement* of $\mathscr{P}_1$ and $\mathscr{P}_2$, that is, the coarsest partition of Ω such that, for each $\omega \in \Omega$,

$$\check{P}(\omega) \subset P_i(\omega), \quad \forall i \in I = \{1, 2\},$$

where $\check{P}(\omega) = P_1 \vee P_2(\omega)$ is the class of the join to which belongs ω.

The classes of the join are obtained by taking for each class of one partition its intersections with the classes of the other partition (see, for instance Barbut 1968).

In the Example from above:

$$\begin{aligned}
\mathscr{P}_1 &= \{\{a,b\},\{c,d\},\{e\},\{f\}\}, \\
\mathscr{P}_2 &= \{\{a,c\},\{b,d\},\{e,f\}\}, \\
\text{The join: } \check{\mathscr{P}} &= \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\}\} \\
\text{Remember, the meet: } \hat{\mathscr{P}} &= \{\{a,b,c,d\},\{e,f\}\}
\end{aligned}$$

A technical note: the matrix representation of two partitions

Any two finite partitions can be written in the form of a matrix such that

- the elements of the matrix are occupied by the elements of the join of the two partitions, with possibly some elements of the matrix empty but without any rows or columns completely empty, and
- the information classes of one individual correspond to the rows of the matrix and that of the other individual to the columns of the matrix (see, for instance, Barbut 1968).

In such a matrix, the classes of the meet of the two partitions appear as the unions of those elements of the join that have the same empty elements along rows as well as columns.

Example:

$$
\begin{aligned}
\mathscr{P}_1 &= \{\{a,b\},\{c,d\},\{e\},\{f\}\},\\
\mathscr{P}_2 &= \{\{a,c\},\{b,d\},\{e,f\}\},\\
\text{the join: } \check{\mathscr{P}} &= \{\{a\},\{b\},\{c\},\{d\},\{e\},\{f\}\}\\
\text{the meet: } \hat{\mathscr{P}} &= \{\{a,b,c,d\},\{e,f\}\}
\end{aligned}
$$

Practical for calculating the posteriors for a certain event A: Let $A = \{b, c\}$ and $\omega = a$ the true state of the world:

$\{a^\star\}$	$\{\mathbf{b}\}$		$\frac{1}{2}$
$\{\mathbf{c}\}$	$\{d\}$		$\frac{1}{2}$
		$\{e\}$	0
		$\{f\}$	0
$\frac{1}{2}$	$\frac{1}{2}$	0	

In the figure above, for each row, to the right of the vertical line (information class of individual 1), appears the conditional probability of A given that row; for each column, below the horizontal line (information class of individual 2), appears the conditional probability of A given the column.

Chapter 4

Bayesian Dialogues

Geanakoplos and Polemarchakis's (1982) scenario of indirect communication

... Imagine that after having received their private information about the true state of the world (according to their information partition), the two individuals, turn in turn, communicate their posteriors back and forth, each round extracting the information that is contained in the announcement of the previous round.

This process is best understood as operating through a successive reduction of the set of possible states of the world:

- The process starts by discarding all states that are not in the information class of the meet to which belongs the true state of the world. Of course, because simply by having received the information through their partitions—thanks to the common knowledge of these partitions—it will be common knowledge between the two individuals that any state that is not in that class of the meet cannot be the true state of the world.
- Then, at each step t, with one of the individuals announcing the posterior probability that he or she attributes to the event of interest A at this step, it becomes common knowledge between the two individuals that a certain subset of Ω at step t cannot contain the true state of the world: namely the union of all those partition classes of the individual who has just announced his or her posterior that do not lead to that posterior. This subset is discarded from Ω at step t to give Ω at step $t+1$.

More formally:

Let $\Omega_0 = \Omega$.
Step 1: $\Omega_1 = \hat{P}(\omega^\star)$, where $\omega^\star$ is the true state of the world.
Step t: $\Omega_t = \Omega_{t-1} \backslash \bar{\mathscr{P}}_{i(t-1),t-1}$, where

$$\bar{\mathscr{P}}_{i(t),t} = \bigcup_{i(t),k} P_{i(t),k}, \text{ such that } \frac{p(A \cap P_{i(t),k} \cap \Omega_t)}{p(P_{i(t),k} \cap \Omega_t)} \neq q_{i(t),t},$$

$$q_{i(t),t} = \frac{p(A \cap P_i(\omega) \cap \Omega_t)}{p(P_i(\omega) \cap \Omega_t)}$$

with $i(t)$ given by the sequence $1, 2, 1, 2, \ldots$ if individual 1 starts, and by $2, 1, 2, 1 \ldots$ if individual 2 starts.

The process ends—more precisely, will have reached an absorbing state—when a subset of Ω is reached such that the announcement of the posterior of any of the two individuals does not allow them to discard any more states.

This terminal subset of Ω will be one on which the “Aumann” conditions hold: the posteriors will be common knowledge—thanks to the common knowledge of the information partitions induced by the reduced set of states of the world at that step—and hence (as Aumann’s result says) will be equal.

A dynamic foundation of Aumann's result

It can be shown that this process converges after a finite number of steps to a situation in which the posteriors are common knowledge and hence—by Aumann's result—identical (Geanakoplos et Polemarchakis 1982). In that sense, such a process can be interpreted as a dynamic foundation of Aumann's result.

One immediate observation: If the Aumann conditions are satisfied (on the original set Ω), the process stops immediately at step 1, or to say it more correctly, will have reached its absorbing state at step 1.

Depends on the order

A Bayesian dialog depends on the order in which the two individuals announce their posteriors (see, for instance, Polemarchakis 2016). Depending on whether it is individual 1 or individual 2 who starts the process, the process can end with *different* subsets of Ω.

Important:

On each of these two different terminal subsets of Ω, the "Aumann" conditions hold. The process, so to say, gets "stopped" by the Aumann conditions. But, on these two different terminal subsets of Ω, different posteriors attributed to A in common knowledge might arise.

An example (in which the order matters)

Derived from an example given by Polemarchakis (2016). Let $\Omega = \{a, b, c, d, e, f, g, h, i, j, k\}$ the set of possible states of the world, endowed with uniform prior probability, that is, $p(\omega) = 1/11$ for all possible states of the world. Furthermore let

$$\begin{aligned} \mathscr{P}_1 &= \{\{a, b, c, d, e, f\}, \{g, h, i, j, k\}\}, \\ \mathscr{P}_2 &= \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}\}, \end{aligned}$$

$A = \{a, b, i, j, k\}$, the event of interest; and $\omega^\star = a$, the true state of the world. Here: $\mathscr{P}_1 \wedge \mathscr{P}_2 = \{\Omega\}$, and $\mathscr{P}_1 \vee \mathscr{P}_2 = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, h\}, \{i, j\}, \{k\}\}$.
In matrix representation:

$$\begin{array}{ccc|c} \{\mathbf{a}^\star, \mathbf{b}\} & \{c, d\} & \{e, f\} & \frac{1}{3} \\ \{g, h\} & \{\mathbf{i}, \mathbf{j}\} & \{\mathbf{k}\} & \frac{3}{5} \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \end{array}$$

In this example, the outcome of a Bayesian dialogue depends on the order in which the two individuals report their posteriors.

If individual 1 starts:

Step 1: $\Omega(1) = \{a, b, c, d, e, f, g, h, i, j, k\}$,
$\mathscr{P}_{1,\Omega(1)} = \{\{a, b, c, d, e, f\}, \{g, h, i, j, k\}\}$

$$q_1 = \frac{p(\{a, b, i, j, k\} \cap \{a, b, c, d, e, f\})}{p(\{a, b, c, d, e, f\})} = \frac{p(\{a, b\})}{p(\{a, b, c, d, e, f\})} = \frac{1}{3}$$

If individual 1 announces $1/3$, then it will become common knowledge between the two individuals that the true state cannot belong to the set $\{g, h, i, j, k\}$, and therefore this set should be deleted from what remains in the *fund of common knowledge*. The matrix becomes:

$\{\mathbf{a}^\star, \mathbf{b}\}$	$\{c, d\}$	$\{e, f\}$	$\frac{1}{3}$
1	0	0	

Step 2: $\Omega(2) = \{a, b, c, d, e, f\}$, $\mathscr{P}_{2,\Omega(2)} = \{\{a, b\}, \{c, d\}, \{e, f\}\}$

$$q_2 = \frac{p(\{a, b\} \cap \{a, b\})}{p(\{a, b\})} = \frac{p(\{a, b\})}{p(\{a, b\})} = 1.$$

If individual 2 announces 1, then it will be common knowledge between the two individuals that the true state of the world cannot be in $\{c, d, e, f\}$, and hence this set can be deleted in common knowledge. The matrix becomes:

$\{\mathbf{a}^\star, \mathbf{b}\}$	1
1	

Step 3: $\Omega(3) = \{a, b\}$, $\mathscr{P}_{1,\Omega(3)} = \{\{a, b\}\}$. Individual 1 announces also "1," and the process has reached its absorbing state. Note that on the set of states that are still alive at step 3, $\Omega(3) = \{a, b\}$, the Aumann conditions are trivially satisfied because the information partitions of the two individuals induced by $\Omega(3) = \{a, b\}$ are identical: $\mathscr{P}_{1,\Omega(3)} = \{\{a, b\}\} = \mathscr{P}_{2,\Omega(3)}$.

In this example, the element of the join to which belongs the true state of the world is also $\{a, b\}$. Direct communication will therefore also lead to a posterior of 1 attributed to A.

But if individual 2 starts:

Step 1: $\Omega(1) = \{a, b, c, d, e, f, g, h, i, j, k\}$,
$\mathscr{P}_{2,\Omega(1)} = \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}\}$

$$q_1 = \frac{p(\{a, b, i, j, k\} \cap \{a, b, g, h\})}{p(\{a, b, g, h\})} = \frac{p(\{a, b\})}{p(\{a, b, g, h\})} = \frac{1}{2}$$

$\rightarrow \{e, f, k\}$ can be deleted in common knowledge. But then the matrix is:

$\{\mathbf{a}^\star, \mathbf{b}\}$	$\{c, d\}$	$\frac{1}{2}$
$\{g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

And the process of deletion ends here, with each of them announcing 1/2 from this moment on, forever.

If individual 1 starts:

Step 1:

$\{\mathbf{a}^\star, \mathbf{b}\}$	$\{c, d\}$	$\{e, f\}$	$\frac{1}{3}$
$\{g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\{\mathbf{k}\}$	$\frac{3}{5}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	

Step 2:

$\{\mathbf{a}^\star, \mathbf{b}\}$	$\{c, d\}$	$\{e, f\}$	$\frac{1}{3}$
1	0	0	

Step 3:

$\{\mathbf{a}^\star, \mathbf{b}\}$	1
1	

If individual 2 starts:

Step 1:

$\{\mathbf{a}^\star, \mathbf{b}\}$	$\{c, d\}$	$\{e, f\}$	$\frac{1}{3}$
$\{g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\{\mathbf{k}\}$	$\frac{3}{5}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	

Step 2:

$\{\mathbf{a}^\star, \mathbf{b}\}$	$\{c, d\}$	$\frac{1}{2}$
$\{g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

Further properties of a Bayesian dialogue

The visible trace of a Bayesian dialogue is the sequence of announced posteriors.

It can be that at level "nothing happens," in the sense that each of the individuals repeats for a certain number of rounds the same posterior, while in the background, nevertheless, the two individuals—in common knowledge—successively discard possible states of the world, namely all those of which it has become common knowledge, up to that step, that they cannot be the true state of the world.

Example (after Aumann; see Geanakoplos et Polemarchakis 1982)

For the general parametric form (for any n) see Geanakoplos et Polemarchakis (1982, 197). Here we see the case $n = 3$.

Soient $\Omega = \{a, b, c, d, e, f, g, h, i\}$ et $p(\omega) = 1/9$ pour tous les événements élémentaires. Supposons que:

$$\begin{aligned} \mathscr{P}_1 &= \{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}\}, \\ \mathscr{P}_2 &= \{\{a, b, c, d\}, \{e, f, g, h\}, \{i\}\}, \end{aligned}$$

$A = \{a, e, i\}$, et $\omega^\star = a$.

Supposons que c'est l'individu 1 qui commence.

A l'étape 1: $\Omega(1) = \{a, b, c, d, e, f, g, h, i\}$,
$\mathscr{P}_{1,\Omega(1)} = \{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}\}$,

$$q_1 = \frac{p(\{a, e, i\} \cap \{a, b, c\})}{p(\{a, b, c\})} = \frac{p(\{a\})}{p(\{a, b, c\})} = \frac{1}{3}$$

Rien ne peut être écarté en connaissance commune.

A l'étape 2: $\Omega(2) = \{a, b, c, d, e, f, g, h, i\}$,
$\mathscr{P}_{2,\Omega(2)} = \{\{a, b, c, d\}, \{e, f, g, h\}, \{i\}\}$,

$$q_2 = \frac{p(\{a, e, i\} \cap \{a, b, c, d\})}{p(\{a, b, c, d\})} = \frac{p(\{a\})}{p(\{a, b, c, d\})} = \frac{1}{4}$$

Cette annonce de l'individu 2 permet d'écarter $\{i\}$ en connaissance commune; puisque $\{i\}$ aurait produit l'annonce $q_2 = 1$.

A l'étape 3: $\Omega(3) = \{a, b, c, d, e, f, g, h\}$,
$\mathscr{P}_{1,\Omega(3)} = \{\{a, b, c\}, \{d, e, f\}, \{g, h\}\}$,

$$q_1 = \frac{p(\{a, e, i\} \cap \{a, b, c\})}{p(\{a, b, c\})} = \frac{p(\{a\})}{p(\{a, b, c\})} = \frac{1}{3}$$

Cette annonce de l'individu 1 permet d'écarter $\{g, h\}$ en connaissance commune; puisque $\{g, h\}$ aurait produit l'annonce $q_1 = 0$.

A l'étape 4: $\Omega(4) = \{a, b, c, d, e, f\}$,
$\mathscr{P}_{2,\Omega(4)} = \{\{a, b, c, d\}, \{e, f\}\}$,

$$q_2 = \frac{p(\{a, e, i\} \cap \{a, b, c, d\})}{p(\{a, b, c, d\})} = \frac{p(\{a\})}{p(\{a, b, c, d\})} = \frac{1}{4}$$

Cette annonce de l'individu 2 permet d'écarter $\{e, f\}$ en connaissance commune; puisque $\{e, f\}$ aurait produit l'annonce $q_2 = 1/2$.

A l'étape 5: $\Omega(5) = \{a, b, c, d\}$, $\mathscr{P}_{1,\Omega(5)} = \{\{a, b, c\}, \{d\}\}$,

$$q_1 = \frac{p(\{a, e, i\} \cap \{a, b, c\})}{p(\{a, b, c\})} = \frac{p(\{a\})}{p(\{a, b, c\})} = \frac{1}{3}$$

Cette annonce de l'individu 1 permet d'écarter $\{d\}$ en connaissance commune; puisque $\{d\}$ aurait produit l'annonce $q_1 = 0$.

A l'étape 6: $\Omega(6) = \{a, b, c\}$, $\mathscr{P}_{2,\Omega(6)} = \{\{a, b, c\}\}$,

$$q_2 = \frac{p(\{a, e, i\} \cap \{a, b, c\})}{p(\{a, b, c\})} = \frac{p(\{a\})}{p(\{a, b, c\})} = \frac{1}{3}$$

A partir de cette étape plus rien ne peut être écarté. Le processus de communication indirecte à travers les croyances a trouvé sa fin – son point fixe. Les deux individus vont à toujours chacun répéter : $1/3$.

La trace visible du processus de communication indirecte, la suite des probabilités actualisées, est:

Etape 1: $q_1 = 1/3$
Etape 2: $q_2 = 1/4$
Etape 3: $q_1 = 1/3$
Etape 4: $q_2 = 1/4$
Etape 5: $q_1 = 1/3$
Etape 6: $q_2 = 1/3$
Etape 7: $q_2 = 1/3$
Etape 8: $q_1 = 1/3$

$\vdots$

Pendant cinq périodes il ne se passe "rien" à la surface des choses: les deux individus répètent chacun ce qu'ils ont dit auparavant, jusqu'à la sixième étape lorsque l'individu 2 annoncera aussi $1/3$, ce qui terminera le processus, c'est-à-dire qu'à partir de ce moment-là ils vont à toujours répéter $1/3$ tous les deux.

In matrix form:

$$
\begin{array}{ccc|c}
\{\mathbf{a}^{\star}, b, c\} & & & \frac{1}{3} \\
\{d\} & \{\mathbf{e}, f\} & & \frac{1}{3} \\
 & \{g, h\} & \{\mathbf{i}\} & \frac{1}{3} \\
\hline
\frac{1}{4} & \frac{1}{4} & 1 &
\end{array}
$$

Any regularities in the sequence of announced posteriors stemming from a Bayesian dialogue?

Polemarchakis (2016) has recently addressed the following question: *Is there any pattern in the sequence of announced probabilities that stem from a Bayesian dialogue?*

Polemarchakis shows that there isn't: that for any sequence of numbers strictly between 0 and 1, $q_1, q_2, q_3, q_4, \ldots, q_N$, one can find a set Ω of possible states of the world and two partitions such that that sequence is the visible trace of a Bayesian, or as Polemarchakis says, a "rational dialogue."

Le résultat est remarquable puisqu'en dehors de la condition que les probabilités sont strictement entre 0 et 1, la suite $q_1, q_2, q_3, q_4, \ldots, q_N$ n'est contrainte par aucune autre condition; notamment aucune condition de monotonie, ni sur les éléments de $q_1, q_2, q_3, q_4, \ldots, q_N$ ni sur les éléments des sous-suites $q_1, q_3, q_5, \ldots,$ ou $q_2, q_4, q_6, \ldots,$.

Ceci a une interprétation forte: juste en écoutant ce que les deux individus se disent – ou bien en examinant le procès-verbal de leur communication – on ne peut pas dire s'ils sont engagés dans un dialogue rationnel (dans le sens bayésien) ou pas: il n'y a rien dans la forme extérieure de ce qu'ils se disent qui nous permettrait de décider si ce qu'ils se disent est rationnel (dans le sens bayésien) ou pas.

La preuve donnée par Polemarchakis est constructive. Elle se sert de l'écriture matricielle des deux partitions d'information.

The following example illustrates Polemarchakis's proof.

Exemple

Soit la suite de probabilités $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{3}$, $q_3 = \frac{1}{4}$, $q_4 = \frac{1}{2}$, $q_5 = \frac{1}{2}$. On cherche un ensemble fondamental Ω, deux partitions de Ω, un événement $A \subset \Omega$ et un état $\omega^\star \in \Omega$ tels que si $\omega^\star$ se réalise, le processus de communication indirecte à la Geanakopolos et Polemarchakis laissera comme trace visible la suite des probabilités ci-dessus, sachant qu'à partir de la sixième étape les deux individus vont à toujours répéter $\frac{1}{2}$.

Supposons que la partition fondue est la plus grossière et la partition croisée la plus fine; c'est-à-dire, tout élément de notre matrice sera occupé par un sous-ensemble de Ω contenant un seul état. En outre nous supposons que tous les états ont a priori la même probabilité de se réaliser. A ce moment, nous ne savons pas encore combien d'états aura Ω; d'autant moins que nous ne savons pas ce qui sera l'événement A. Ces deux éléments se décideront en fonction de notre construction.

En ce qui concerne la représentation donnée ici, les états différents se distinguent tout simplement par leur emplacement dans la matrice. Un état est représenté par le symbole $\circ$ s'il n'appartient pas à l'événement A, et par $\bullet$ s'il appartient à l'événement A. Suivant Polemarchakis, nous supposons que l'état réalisé est celui au croisement de la première ligne et de la première colonne.

On commence par la fin. A la fin on veut que les deux individus disent $1/2$. Voici une matrice qui satisfait cette condition:

●	○	$\frac{1}{2}$
○	●	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

$$1 \text{ dit} : q_5 = 1/2$$
$$2 \text{ dit} : q_4 = 1/2$$

C'est bien sûr une situation Aumannienne. Remarquons que notre construction n'est pas unique – ce qui ne gêne pas puisque nous chercherons à démontrer l'existence et non l'unicité d'un certain objet.

Ensuite, on veut que l'individu 1, à étape 3, avant que le processus ne trouve sa fin avec les conditions d'Aumann, ait dit $1/4$. Comment peut-on élargir la matrice pour arriver à cette fin? Une possibilité est de rajouter tout simplement deux états n'appartenant pas à A (deux petits cercles vides) à chaque ligne:

$$\begin{array}{cccc|c} \bullet & \circ & \circ & \circ & \frac{1}{4} \\ \circ & \bullet & \circ & \circ & \frac{1}{4} \\ \hline \frac{1}{2} & \frac{1}{2} & 0 & 0 & \end{array} \qquad 1 \text{ dit} : q_3 = 1/4$$

Ensuite, on veut que l'individu 2, à l'étape précédente, ait dit $1/3$. Voici une extension de la matrice qui fournit ce résultat:

$$\begin{array}{cccc|c} \bullet & \circ & \circ & \circ & \frac{1}{4} \\ \circ & \bullet & \circ & \circ & \frac{1}{4} \\ \circ & \circ & \bullet & \bullet & \frac{1}{2} \\ \hline \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \end{array} \qquad 2 \text{ dit} : q_2 = 1/3$$

Et finalement on veut que tout ait commencé avec l'individu 1 qui dit 1/2. Voici une matrice qui le traduit:

●	○	○	○	●	●	$\frac{1}{2}$
○	●	○	○	●	●	$\frac{1}{2}$
○	○	●	●	○	●	$\frac{1}{2}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	

$1 \text{ dit} : q_1 = 1/2$

Pour faire le test, il faut remonter le fil de l'argument; il faut commencer avec la matrice tout en bas et appliquer l'algorithme de la communication indirecte.

Hors compétition

AN EXPERIMENT

An experiment

Let $\Omega = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q\}$ the set of possible states of the world, endowed with uniform prior probability, that is, $p(\omega) = 1/17$ for all possible states of the world. Furthermore let

$$\begin{aligned}
\mathscr{P}_1 &= \{\{a, b, c, d, e, f\}, \{g, h, i, j, k, l\}, \{m, n, o, p, q\}\}, \\
\mathscr{P}_2 &= \{\{a, b, c, g, h, i, m, n, o\}, \{d, e, f, j, k, l, p, q\}\},
\end{aligned}$$

$A = \{a, b, c, j, k, l, p, q\}$, the event of interest.

In matrix representation:

$$\begin{array}{cc|c}
\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} & \{d, e, f\} & \\
\{g, h, i\} & \{\mathbf{j}, \mathbf{k}, \mathbf{l}\} & \\
\{m, n, o\} & \{\mathbf{p}, \mathbf{q}\} & \\
\hline
 & &
\end{array}$$

In class, December 3rd, 2020, we played this game

Participation was voluntary. There was a real payoff to be won. Students participating in the experiment could win up to 4 points that would be added to the grade for participation in this class, which is based on a system of 20 possible points (10 for participation and 10 for the final essay).

The story was translated into real payoffs in the following way:

Each student who participates in the experiment gets an initial endowment of 2 points. If the player who is asked in the third round gives the correct answer to the question whether A happend, then both get 4 points; if the answer is wrong, they both lose their endowment of the 2 points (that is, they have earned nothing for participating in the experiment). There is however also an outside option: the player who is asked in the third round can say I do not want to give an answer. If he or she does so, then both students leave the experiment with their endowment of 2 points. Points from the experiment are added to the points for participaiton in class.

What happened:

$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$	$\{d, e, f\}$
$\{g, h, i\}$	$\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$
$\{m, n, o\}$	$\{\mathbf{p}, \mathbf{q}\}$

Game 1:
Individual 1: $q_1 = 1/2$
Individual 2: $q_2 = (3/8)\,3/9$
Individual 1: $q_1 = 1$

Game 2:
Individual 1: $q_1 = 1/2$
Individual 2: $q_2 = 1/3$
Individual 1: $q_1 = 1$

Game 3:
Individual 1: $q_1 = 1/2$
Individual 2: $q_2 = (1/3)\,9/17$
Individual 1: stop.

Game 4:
Individual 2: $q_2 = 3/8$
Individual 1: $q_1 = 1/2$
Individual 2: stop

Game 5:
Individual 2: $q_2 = 3/9$
Individual 1: $q_1 = 1/2$
Individual 2: $q_1 = 1$

Observations

None of these plays of the game was according to the definition of a Bayesian dialogue!

The solution according to a Bayesian dialogue

If individual 1 starts:

$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$	$\{d, e, f\}$	$\frac{1}{2}$
$\{g, h, i\}$	$\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$	$\frac{1}{2}$
$\{m, n, o\}$	$\{\mathbf{p}, \mathbf{q}\}$	$\frac{2}{5}$
$\frac{3}{9}$	$\frac{5}{8}$	

If 1 says $\frac{1}{2}$ $\rightarrow$ common knowledge between the two that the true state cannot belong to the last row $\rightarrow$ delete last row

$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$	$\{d, e, f\}$	$\frac{1}{2}$
$\{g, h, i\}$	$\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	

They would be caught in the Aumann conditions!

- In two out of the three plays of the game in which player 1 talked first, the player who talked second, rather than extracting information from the announcment of the player who talked first, announced his posterior based on the information that he had received from the director as a function of his information partition.
- This allowed them to escape being trapped in the Aumann conditions!
- So it was rational to do so!

If individual 2 starts:

$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$	$\{d, e, f\}$	$\frac{1}{2}$
$\{g, h, i\}$	$\{\mathbf{j}, \mathbf{k}, \mathbf{l}\}$	$\frac{1}{2}$
$\{m, n, o\}$	$\{\mathbf{p}, \mathbf{q}\}$	$\frac{2}{5}$
$\frac{3}{9}$	$\frac{5}{8}$	

If 2 says $\frac{3}{9} \rightarrow$

$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$	1
$\{g, h, i\}$	0
$\{m, n, o\}$	0
$\frac{3}{9}$	

1 can already annonce that the probability of A is equal to 1!

- This is not what happend in the two plays of the game in which individual 2 talked first that we observed (tournnaments 4 and 5): in tournament 4, the individual who talked second (individual 1) did not announce ‘1’ but the posterior given the information that he had received from the director. Still, because of the structrue of the game, this let individual 2 know that 1 had received the informaiton that the true state belong to the first column and he could confidently say: 'Yes, event A did happen.'
- As theorists we could ask: In case that individual 2 starts, how would have the participants in the game played the game if the rules of the game had been that already at step 2 individual 1 has to say whether A happened or not?

Chapter 5

Indirect communication through acts (bets)

Imagine that someone is willing to bet 1000 euros, at a rate $1:1$, that a certain candidate wins a competition. It is as if the person were publicly saying that he or she attributes a probability of at least 50% to the event that candidate in question willl win—assuming that the person is risk neutral and want to maximize his or her monetary gain.

... This information might allow another person to update the probability that he or she thinks should be attributed to the event that the candidate in question will win, which might influence whether he or she want to take such a bet. If the first observes that decsion of the second, that might in turn allow the first to update his or her probability that candidate in question will win; etc.

If the two individuals are commonly aware of this kind of interaction, they find themselves in a Bayesian dialouge mediated by their actions in the betting market.

Within the formal framework that we have studied here, assuming that the information partitions of the two individuals are common knowledge: if i accepts the bet, it will allow the two individuals to discard any states from the set of possible states of the world Ω that belong to an information class of i's partition which leads to a posterior probability that the candidate in question will win of less than 50%.

Sebenius et Geanakoplos (1983) study such a process in more detail. They show that after a finite number of rounds (assuming finite information partitions)will end in a situaiton where one of the two individuals will refuse to take the bet.

Milgrom et Stocky (1982) study a similar process for a market.

Exemple

To illustrate the process studied by Sebenius and Geanakoplos (1983): all possible states of the world have the same prior probability. If the event A happened, individual 2 has to pay 1 the sum of 1000 euros; if A did not happen, 1 has to pay to 2 the sum of 1000 euros:

$\{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\}$	$\{e, f, g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\{\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}\}$	$\frac{9}{14}$
$\{o, p, q, r\}$	$\{\mathbf{s}, \mathbf{t}, \mathbf{u}, v\}$	$\{w, x, y, z\}$	$\{a', b'\}$	$\frac{3}{14}$
$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{6}$	$\frac{4}{6}$	

If individual 1 is asked first whether she wants to take the bet, she says yes, because she attributes to A a probability of 9/14. If this happens in front of individual 2:

$\{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\}$	$\{e, f, g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\{\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}\}$	$\frac{9}{14}$
$\frac{3}{4}$	0	1	1	

If then individual 2 is asked whether he wants to take the bet, he will refuse, which will make it common knowledge between the two that the true state of the world cannot belong to $\{e, f, g, h\}$:

$\{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\{\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}\}$	$\frac{9}{10}$
$\frac{3}{4}$	1	1	

Depends also on the order

Exemple

If individual 2 is asked first whether he want to take the bet or not:

$$
\begin{array}{ccc|c}
\{\mathbf{a}^{\star}, \mathbf{b}, \mathbf{c}, d\} & \{e, f, g, h\} & \{\mathbf{i}, \mathbf{j}\} & \frac{5}{10} \\
\{o, p, q, r\} & \{\mathbf{s}, \mathbf{t}, \mathbf{u}, v\} & \{w, x, y, z\} & \frac{3}{12} \\
\hline
\frac{3}{8} & \frac{3}{8} & \frac{2}{6} &
\end{array}
$$

If then 1 is asked, it is possible that she accepts (but nor sure):

If she accepts:

$\{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\}$	$\{e, f, g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\frac{5}{10}$
$\frac{3}{4}$	0	1	

At this point, individual 2 will refuse the bet, which will make it common knowledge between the two that the true state cannot be outside the set $\{a, b, c, d, i, j\}$:

$\{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\frac{5}{6}$
$\frac{3}{4}$	1	

Communication through acts is not less powerful

Exemple

The same example as above, but according to the process of indirect communication through the exact values of the posteriors (after Geanakoplos et Polemarchakis 1982):

$\{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\}$	$\{e, f, g, h\}$	$\{\mathbf{i}, \mathbf{j}\}$	$\{\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}\}$	$\frac{9}{14}$
$\{o, p, q, r\}$	$\{\mathbf{s}, \mathbf{t}, \mathbf{u}, v\}$	$\{w, x, y, z\}$	$\{a', b'\}$	$\frac{3}{14}$
$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{6}$	$\frac{4}{6}$	

If individual 1 starts:

$$\begin{array}{cccc|c} \{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\} & \{e, f, g, h\} & \{\mathbf{i}, \mathbf{j}\} & \{\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}\} & \frac{9}{14} \\ \hline \frac{3}{4} & 0 & 1 & 1 & \end{array}$$

And 2 is asked then:

$$\begin{array}{c|c} \{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\} & \frac{3}{4} \\ \hline \frac{3}{4} & \end{array},$$

If individual 2 starts:

$$\begin{array}{cc|c} \{\mathbf{a}^\star, \mathbf{b}, \mathbf{c}, d\} & \{e, f, g, h\} & \frac{3}{8} \\ \{o, p, q, r\} & \{\mathbf{s}, \mathbf{t}, \mathbf{u}, v\} & \frac{3}{8} \\ \hline \frac{3}{8} & \frac{3}{8} & \end{array}$$

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