Beliefs "off the equilibrium path," the Index of equilibria, and Invariance

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Based partly on joint work Josef Hofbauer

Probabilities and Interaction

December 21, 2023

What game theorists do ..

Definition of a game	Solution concepts
Game in normal form (matrix)	Nash equilibrium
Game in extensive form (tree)	sequential Bayesian Nash equilibrium

- Extend existing solution concepts to more general classs of games
- Often: solution not unique; multipicity of equilibria $\longrightarrow$ "refine" solution concepts

Signaling games

Games of incomplete information with an explicit sequential structure, given by a game tree

- Sequential Bayesian Nash equilibrium (Kreps and Wilson 1982):

  Profile of strategies and vector of beliefs for every information set, such that:

  – at every information set, player acting there chooses a best response, given his beliefs (probability assessment) over states of Nature and other players' choices,

  – along the path through the game induced by this profile of strategies, beliefs are compatible with Bayes' Law.

- Problem: Often many sequential Bayesian Nash equilibria.

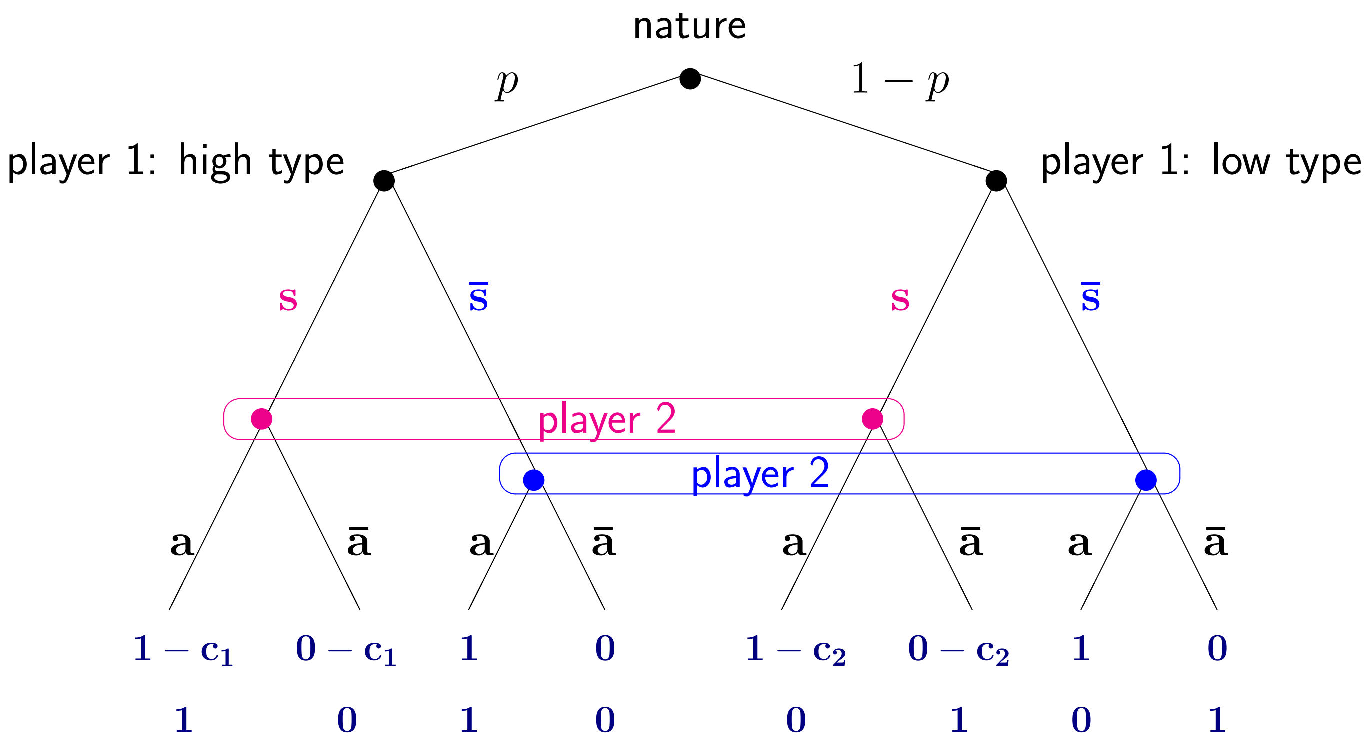
  "Off the equilibrium path": Bayes' Law is not defined. Shall we impose restrictions there?

In this talk

$\longrightarrow$ 3 approaches to "refine" sequential Bayesian Nash equilibrium:

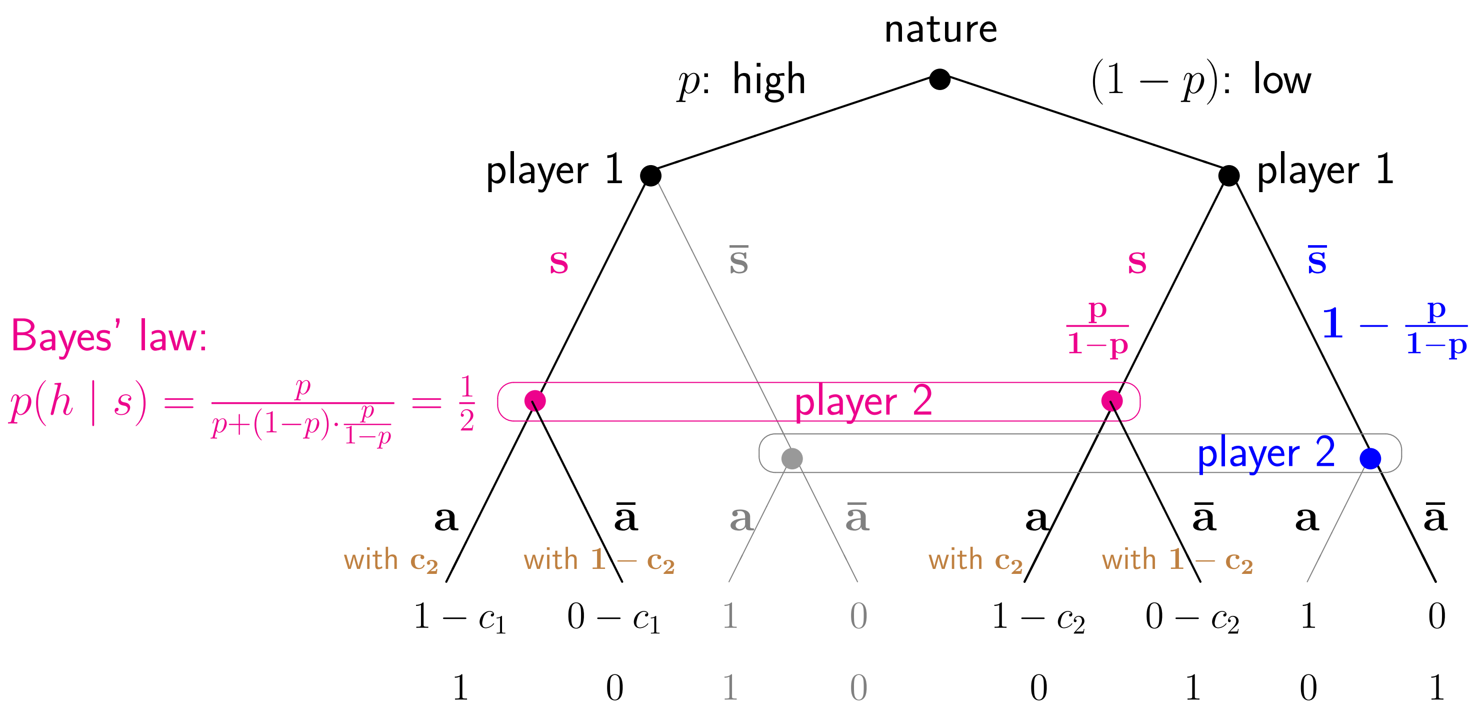
- Impose restrictions on beliefs "off the equilibrium path"
- Index of equilibria (topological properties of associated fixed-point)
- Invariance: requirement that Nash equilibrium in normal form corresponds to a sequential Bayesian Nash equilibrium in every extensive-form game that maps to the same normal-form game.

Costly-signaling game (discrete version of Spence 1973)



	aa	**aā**	**āa**	**āā**
ss	$1 - pc_1 - (1-p)c_2,\ p$	$1 - pc_1 - (1-p)c_2,\ p$	$-pc_1 - (1-p)c_2,\ 1-p$	$-pc_1 - (1-p)c_2,\ 1-p$
ss̄	$1 - pc_1,\ p$	$p(1-c_1),\ 1$	$-pc_1 + (1-p),\ 0$	$-pc_1,\ 1-p$
s̄s	$1 - (1-p)c_2,\ p$	$(1-p)(1-c_2),\ 0$	$p - (1-p)c_2,\ 1$	$-(1-p)c_2,\ 1-p$
s̄s̄	$1,\ p$	$0,\ 1-p$	$1,\ p$	$0,\ 1-p$

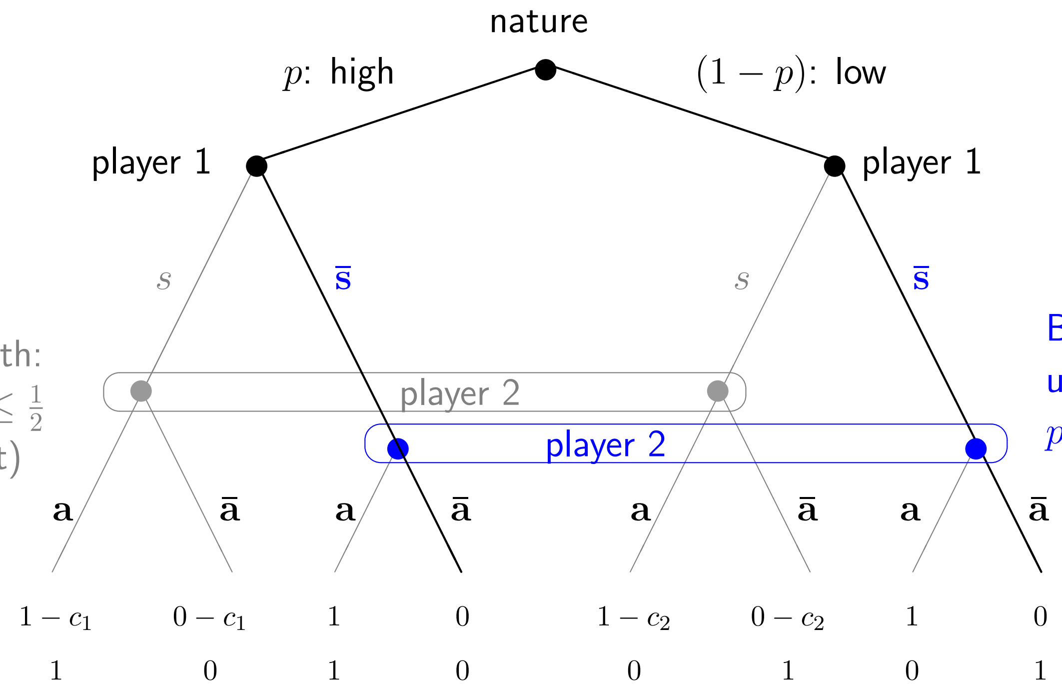
Case $0 \leq c_1 < c_2 < 1$, $p < 1/2$: E1 partially revealing equilibrium



		$\mathbf{aa}$	$\mathbf{c_2} \downarrow$ $\mathbf{a\bar{a}}$	$\mathbf{\bar{a}a}$	$\mathbf{1-c_2} \downarrow$ $\mathbf{\bar{a}\bar{a}}$
$\frac{\mathbf{p}}{\mathbf{1-p}} \rightarrow$	$\mathbf{ss}$	$1-pc_1-(1-p)c_2,\ p$	$\mathbf{1-pc_1-(1-p)c_2},\ p$	$-pc_1-(1-p)c_2,\ \mathbf{1-p}$	$-pc_1-(1-p)c_2,\ \mathbf{1-p}$
$\mathbf{1} - \frac{\mathbf{p}}{\mathbf{1-p}} \rightarrow$	$\mathbf{s\bar{s}}$	$1-pc_1,\ p$	$p(1-c_1),\ \mathbf{1}$	$-pc_1+(1-p),\ 0$	$-pc_1,\ 1-p$
	$\mathbf{\bar{s}s}$	$1-(1-p)c_2,\ p$	$(1-p)(1-c_2),\ 0$	$p-(1-p)c_2,\ \mathbf{1}$	$-(1-p)c_2,\ 1-p$
	$\mathbf{\bar{s}\bar{s}}$	$\mathbf{1},\ p$	$0,\ \mathbf{1-p}$	$\mathbf{1},\ p$	$\mathbf{0},\ \mathbf{1-p}$

- E1: 1 mixes between ss and $s\bar{s}$ with $\frac{p}{1-p}$ on first; 2 between $a\bar{a}$ and $\bar{a}\bar{a}$, with c_2 on first.

Case $0 \le c_1 < c_2 < 1$, $p < 1/2$: P1 "no-signaling" equilibrium outcome



Off equilibrium path:
must have belief $\le \frac{1}{2}$
(a with c_1 at most)
"Plausible"?

By Bayes' law,
updated belief = prior:
$p(h \mid \bar{s}) = p < \frac{1}{2}$

	$\mathbf{aa}$	with $y \in [0, c_1] \to \mathbf{a\bar{a}}$	$\mathbf{\bar{a}a}$	with $1-y \to \mathbf{\bar{a}\bar{a}}$
$\mathbf{ss}$	$1-pc_1-(1-p)c_2,\ p$	$\mathbf{1-pc_1-(1-p)c_2},\ p$	$-pc_1-(1-p)c_2,\ \mathbf{1-p}$	$-pc_1-(1-p)c_2,\ \mathbf{1-p}$
$\mathbf{s\bar{s}}$	$1-pc_1,\ p$	$p(1-c_1),\ \mathbf{1}$	$-pc_1+(1-p),\ 0$	$-pc_1,\ 1-p$
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$\mathbf{\bar{s}\bar{s}}$	$\mathbf{1},\ p$	$0,\ \mathbf{1-p}$	$\mathbf{1},\ p$	$\mathbf{0},\ \mathbf{1-p}$

- P1: No-signaling: 1 takes $\bar{s}\bar{s}$; 2 mix between $a\bar{a}$ and $\bar{a}\bar{a}$ with $y \in [0, c_1]$ on first.

Table 1. Equilibrium structure of the game in Figure 1: $0 \leq c_1 < c_2 < 1$

Prior	Equilibrium component	Index	Rep. dyn.	BR dyn.	NWBR, forward induction	Invariance criterion	Payoffs:
$p < \frac{1}{2}$	(E1): *partially revealing/ partially pooling in* s*:* $(1, \frac{p}{1-p}, c_2, 0)$	$+1$	stable	as. stable	yes	invariant	$h : c_2 - c_1$ $\ell : 0$ $2 : 1 - p$
	(P1): *pooling in* $\bar{s}$*:* $(0, 0, y, 0)$, $y \in [0, c_1]$	0	unstable	unstable	no	not invariant	h: 0 ℓ: 0 $2 : 1 - p$
$p > \frac{1}{2}$	(E2): *partially revealing/ partially pooling in* $\bar{s}$*:* $(1 - \frac{1-p}{p}, 0, 1, 1 - c_1)$	-1	unstable	unstable	yes	invariant	h: $1 - c_1$ ℓ: $1 - c_1$ $2 : p$
	(P2): *pooling in* s*:* $(1, 1, 1, y')$, $y' \in [0, 1 - c_2]$	$+1$	stable	as. stable	yes	invariant	h: $1 - c_1$ ℓ: $1 - c_2$ $2 : p$
	(P3): *pooling in* $\bar{s}$*:* $(0, 0, y, 1)$, $y \in [0, 1]$	$+1$	as. stable	as. stable	yes	not invariant	h: 1 ℓ: 1 $2 : p$
$p = \frac{1}{2}$	(E1'-P2): *pooling in* s*:* $(1, 1, y, y')$, $y \in [c_2, 1]$, $y' \in [0, y - c_2]$)	$+1$	stable	as. stable	yes	invariant	h: $[c_2 - c_1, 1 - c_1]$ ℓ: $[0, 1 - c_2]$ $2 : \frac{1}{2}$
	(P1-E2'-P3): *pooling in* $\bar{s}$*:* $(0, 0, y, y')$, $(y, y') \in [0, 1]^2$, $y \leq y' + c_1$	0	unstable	unstable	only when $y' \in [1-c_1, 1]$	not invariant	h: $[0, 1]$ ℓ: $[0, 1]$ $2 : \frac{1}{2}$

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	(P1): *pooling in* $\bar{s}$*:* $(0, 0, y, 0)$, $y \in [0, c_1]$	0	unstable	unstable	no	not invariant	h: 0 ℓ: 0 $2 : 1 - p$
$p > \frac{1}{2}$	(E2): *partially revealing/ partially pooling in* $\bar{s}$*:* $(1 - \frac{1-p}{p}, 0, 1, 1 - c_1)$	-1	unstable	unstable	yes	invariant	h: $1 - c_1$ ℓ: $1 - c_1$ $2 : p$
	(P2): *pooling in* s*:* $(1, 1, 1, y')$, $y' \in [0, 1 - c_2]$	$+1$	stable	as. stable	yes	invariant	h: $1 - c_1$ ℓ: $1 - c_2$ $2 : p$
	(P3): *pooling in* $\bar{s}$*:* $(0, 0, y, 1)$, $y \in [0, 1]$	$+1$	as. stable	as. stable	yes	not invariant	h: 1 ℓ: 1 $2 : p$

1) The index of equilibria

Shapley (1974): Index, $+1$ or -1, to every regular equilibrium

- Strict equilibrium has index $+1$.
- Removing or adding unused strategies does not change the index.
- *Index Theorem*: the sum of the indices of all equilibria is $+1$.

Hofbauer and Sigmund (1988, 1998): index as the sign of the determinant of the negative Jacobian of the replicator dynamics

Ritzberger (1994, 2002): extends this to equilibrium components:

- Index as an integer, such that the sum over all components is again $+1$
- Robust under payoff perturbations: Let C be a component and U an open neighborhood of C such that all equilibria in the closure of U are already in C. Let C^ε be the set of all equilibria of the perturbed game that lie in U—the finite union of connected components $C_1^\varepsilon, \ldots, C_k^\varepsilon$. By Brouwer's degree theory, the sum of the indices of $C_1^\varepsilon, \ldots, C_k^\varepsilon$ equals the index of C. (C^ε might be empty—but only if C has index 0.)

Demichelis and Ritzberger (2003):

- If an equilibrium component is asymptotically stable under some evolutionary dynamics, then its index equals its Euler characteristics.

  If it is convex or contractible, then its index is $+1$.

In our game (based on Hofbauer and Pawlowitsch 2023):

$p < 1/2$:

- E1: Isolated and quasistrict $\longrightarrow$ regular
  - removing unused strategies $\longrightarrow 2 \times 2$ cyclic game
  - in this game, E1 only equilibrium $\longrightarrow$ index $+1$

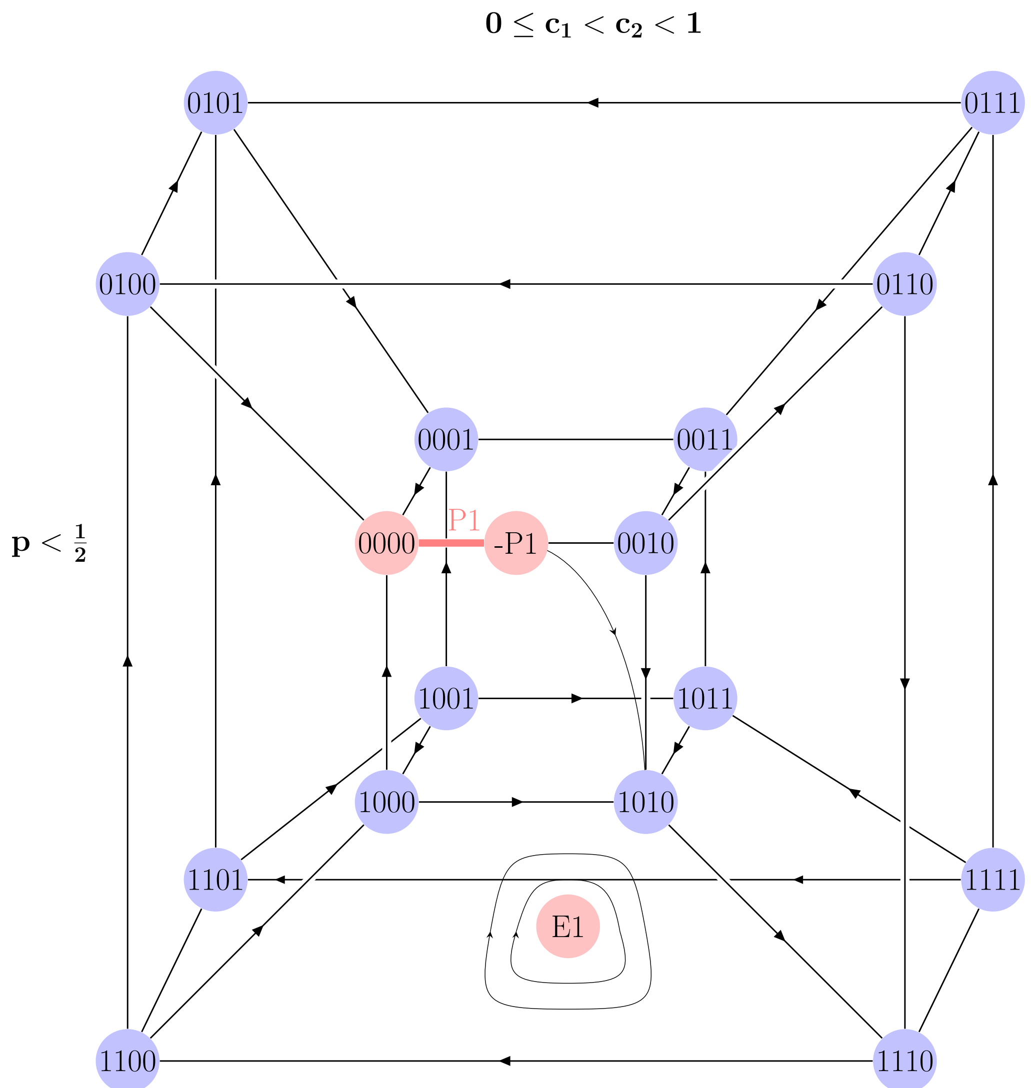
  $\Rightarrow$ candidate for asymptotically stable equilibrium
- P1: by Index Theorem $\longrightarrow$ index 0

  $\Rightarrow$ not asymptotically stable, under no evolutionary dynamics

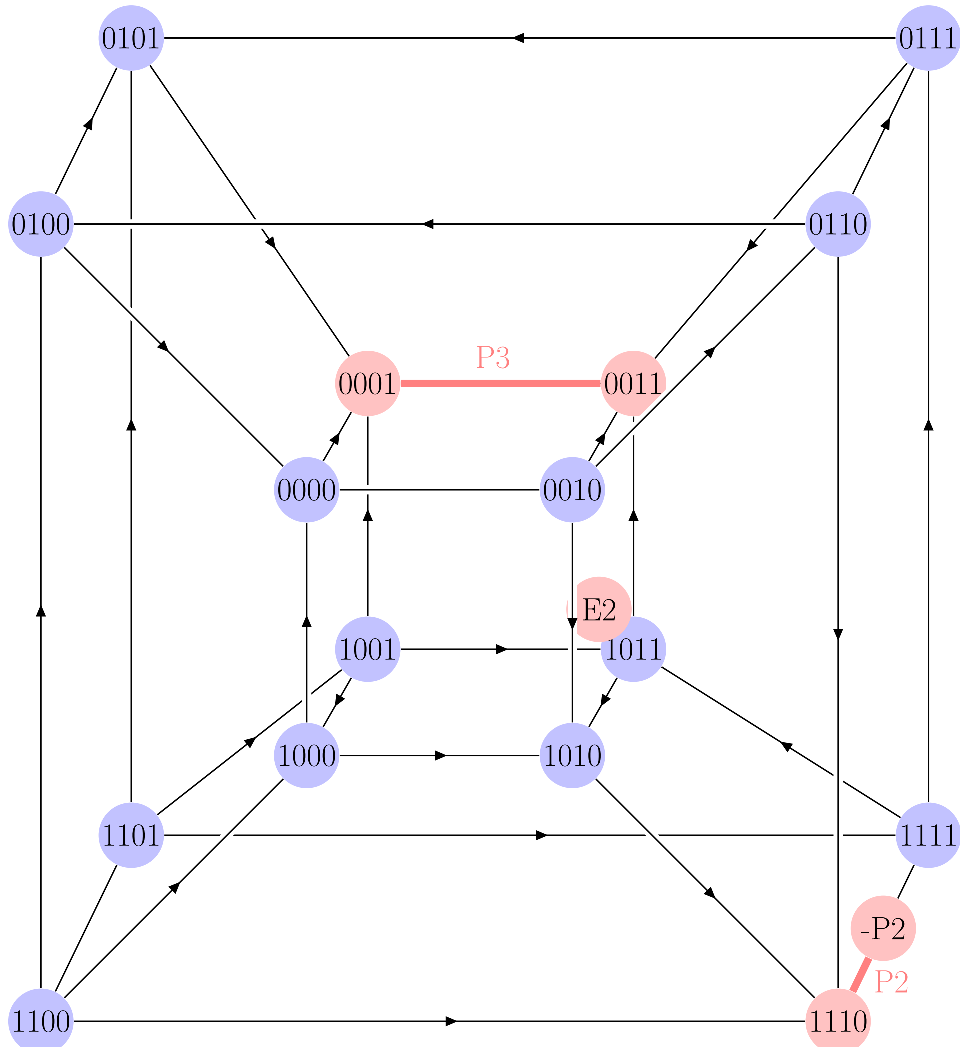
$p > 1/2$:

- P2: by robustness $\longrightarrow$ index $+1$
- E2: Isolated and quasistrict $\longrightarrow$ regular
  - removing unused strategies $\longrightarrow 2\times2$ coordination game with 3 equilibria: E2 and two strict equilibria (index $+1$)
  - by Index Theorem $\longrightarrow$ index -1.
- P3: by Index Theorem $\longrightarrow$ index $+1$

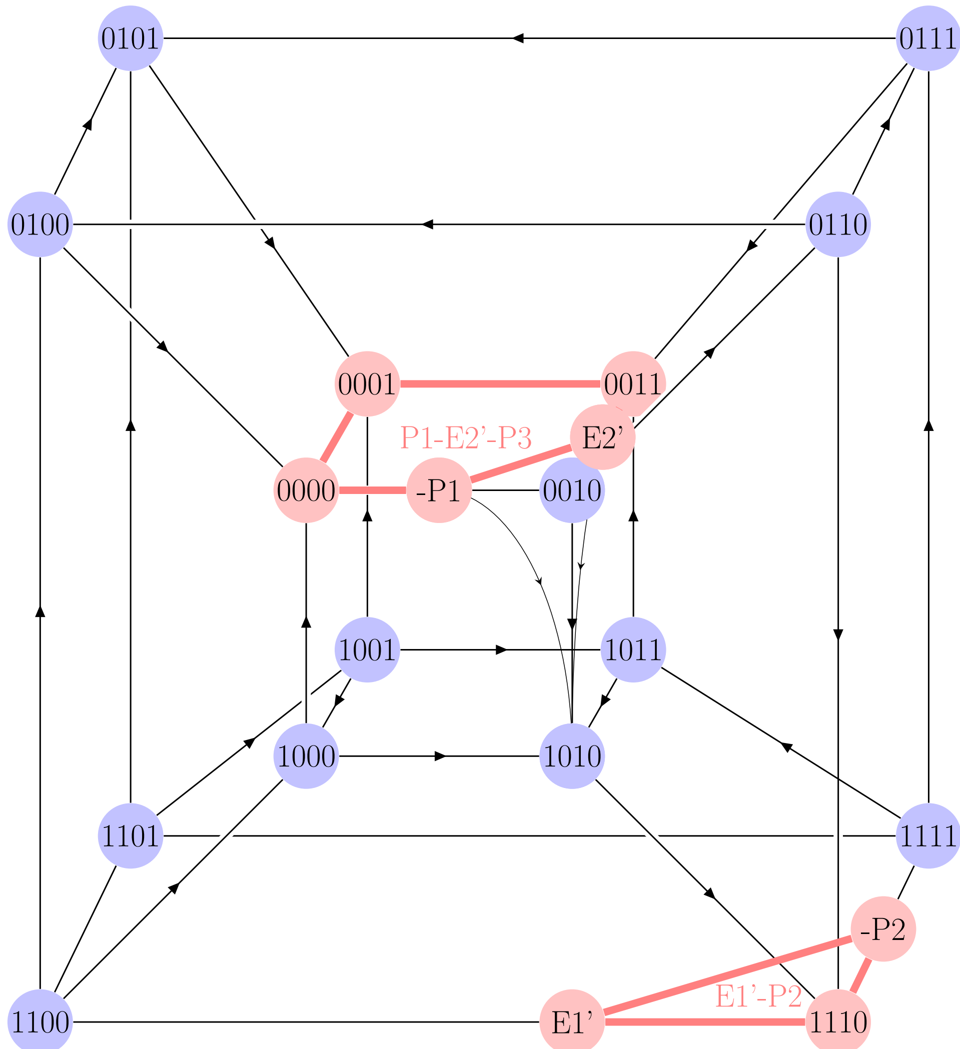
Replicator dynamics

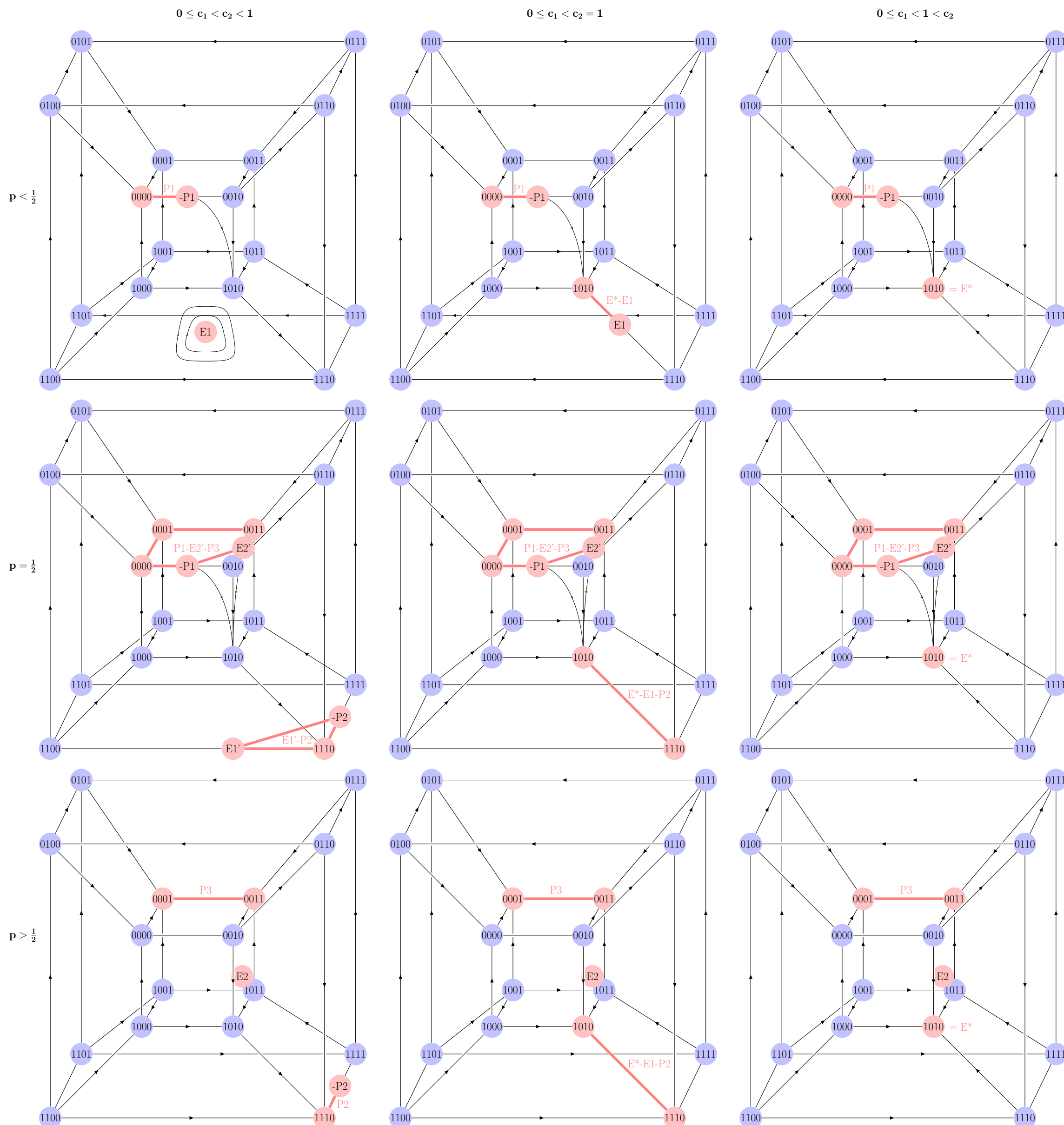


$\mathbf{p} > \frac{1}{2}$



$\mathbf{p} = \frac{1}{2}$





$0 \le c_1 < c_2 < 1$
$0 \le c_1 < c_2 = 1$
$0 \le c_1 < 1 < c_2$
$p < \frac{1}{2}$
$p = \frac{1}{2}$
$p > \frac{1}{2}$
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0001
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1101
1110
1111
P1
-P1
E1
E*-E1
= E*
P1-E2'-P3
E2'
E1'
E1'-P2
-P2
E*-E1-P2
P3
E2
P2

2) Restricting beliefs "off the equilibrium path"

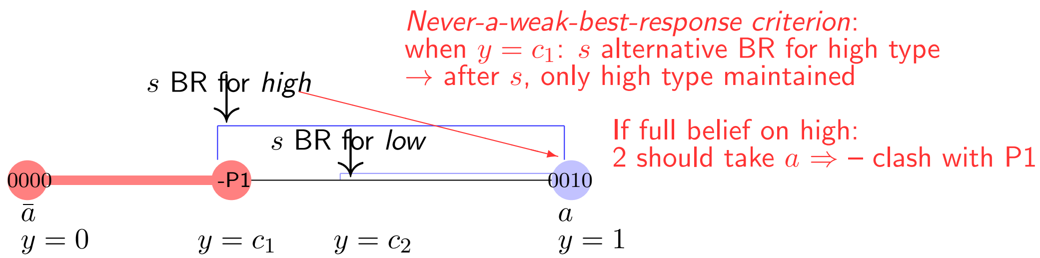
In signaling games: "off the equilibrium path" = after an unused signal

- Cho and Kreps (1987): "never-a-weak-best-response" criterion
- Banks and Sobel (1987): "divinity"
- Govindan and Wilson (2009): "forward induction"

$\rightarrow$ all coincide here. Quite weak selection force: discard the no-signaling equilibrium outcome P1; all other equilibria survive (for the two generic cases $p < 1/2$ and $p > 1/2$).

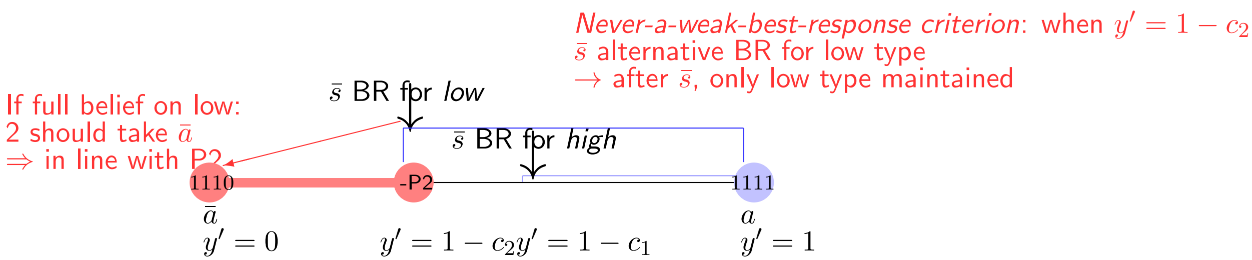
$p < 1/2:$

P1 ($\bar{s}\bar{s} \rightarrow \bar{a}$): NOT robust against "belief-based" refinements: responses of player 2 to the off-the-equilibrium-path signal s:



$p > 1/2:$

P2 ($ss \rightarrow a$): robust against "belief-based" refinements: responses of player 2 to the off-the-equilibrium-path signal $\bar{\mathrm{s}}$:



3) Invariance

Kohlberg and Mertens (1986) : a Nash equilibrium should be selected only if it corresponds to a sequential Bayesian Nash equilibrium in every extensive-form game that maps to the same (reduced) normal form.

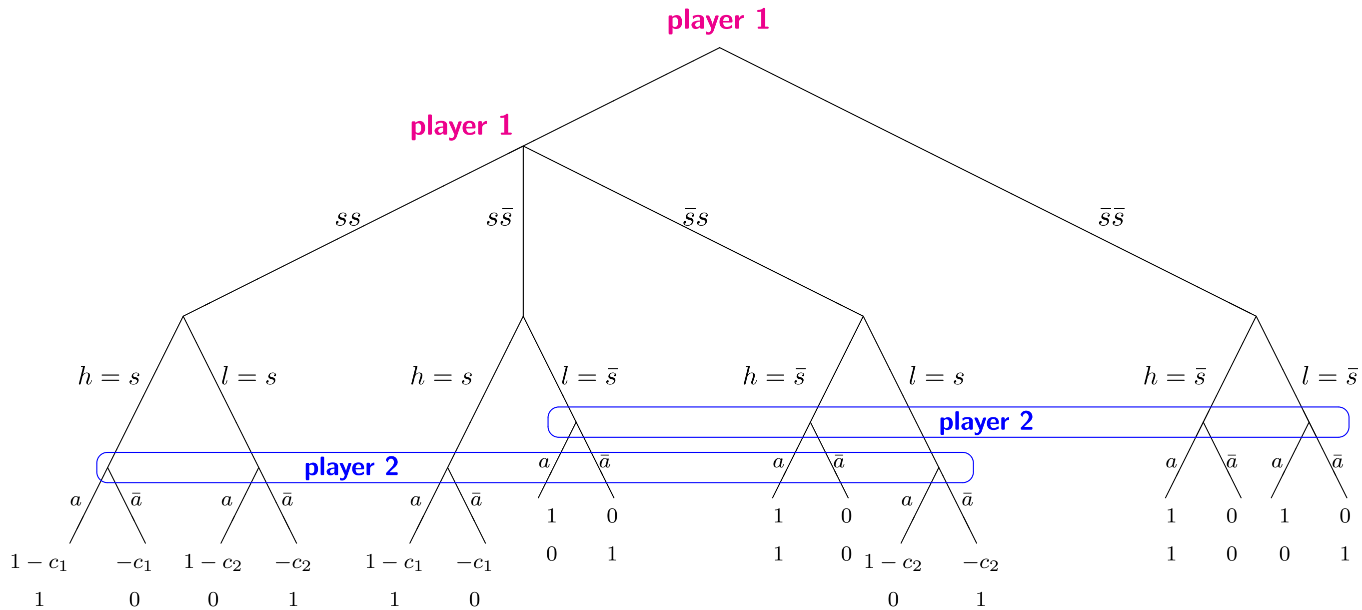
Govindan and Wilson (2009):

invariance $\Rightarrow$ forward induction $\Rightarrow$ never-a-weak-best-response, “divinity”

For the game studied here:

invariance $\Rightarrow$ forward induction $\Leftrightarrow$ never-a-weak-best-response, “divinity”

$\mathbf{p < 1/2}$: P1 (index 0) not forward induction $\Rightarrow$ not invariant



	aa	**a$\bar{\text{a}}$**	**$\bar{\text{a}}$a**	**$\bar{\text{a}}\bar{\text{a}}$**
ss	$1 - pc_1 - (1-p)c_2,\ p$	$1 - pc_1 - (1-p)c_2,\ p$	$-pc_1 - (1-p)c_2,\ 1-p$	$-pc_1 - (1-p)c_2,\ 1-p$
s$\bar{\text{s}}$	$1 - pc_1,\ p$	$p(1-c_1),\ 1$	$-pc_1 + (1-p),\ 0$	$-pc_1,\ 1-p$
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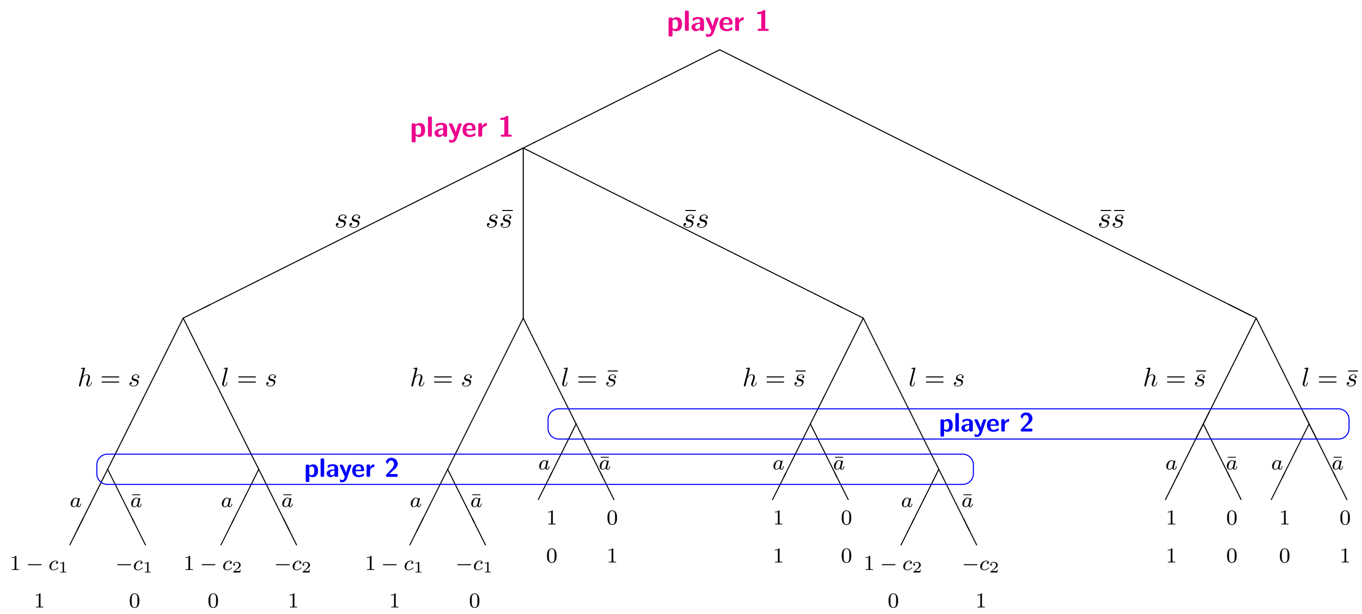
$\mathbf{p < 1/2}$: P1 (index 0) not forward induction $\Rightarrow$ not invariant



	$\mathbf{aa}$	$\mathbf{a\bar{a}}$	$\mathbf{\bar{a}a}$	$\mathbf{\bar{a}\bar{a}}$
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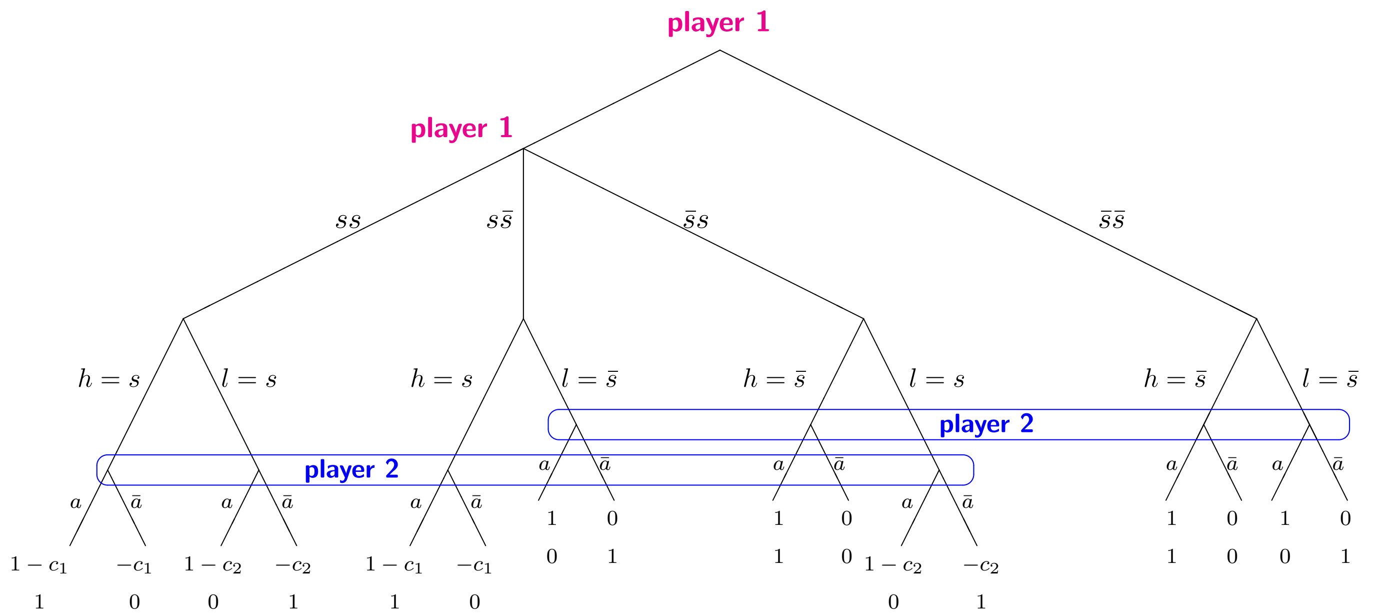
	$\mathbf{aa}$	$\mathbf{a\bar{a}}$	$\mathbf{\bar{a}a}$	$\mathbf{\bar{a}\bar{a}}$
$\mathbf{\bar{s}\bar{s}}$	$1,\ p$	$0,\ 1-p$	$1,\ p$	$0,\ 1-p$

Case $\mathbf{p > 1/2}$, $2p - 1 < (1-p)c_2 - pc_1$, which guarantees that $\bar{s}s$ (high $\bar{s}$, low s) is strictly dominated by $s\bar{s}$: equilibrium outcome P3, in which both types of player 1 use $\bar{s}$ and player 2 accepts (a) (and can have any reaction to s) has index $+1$ and satisfies forward induction ... Is it a sequential equilibrium in the following extensive-form game?



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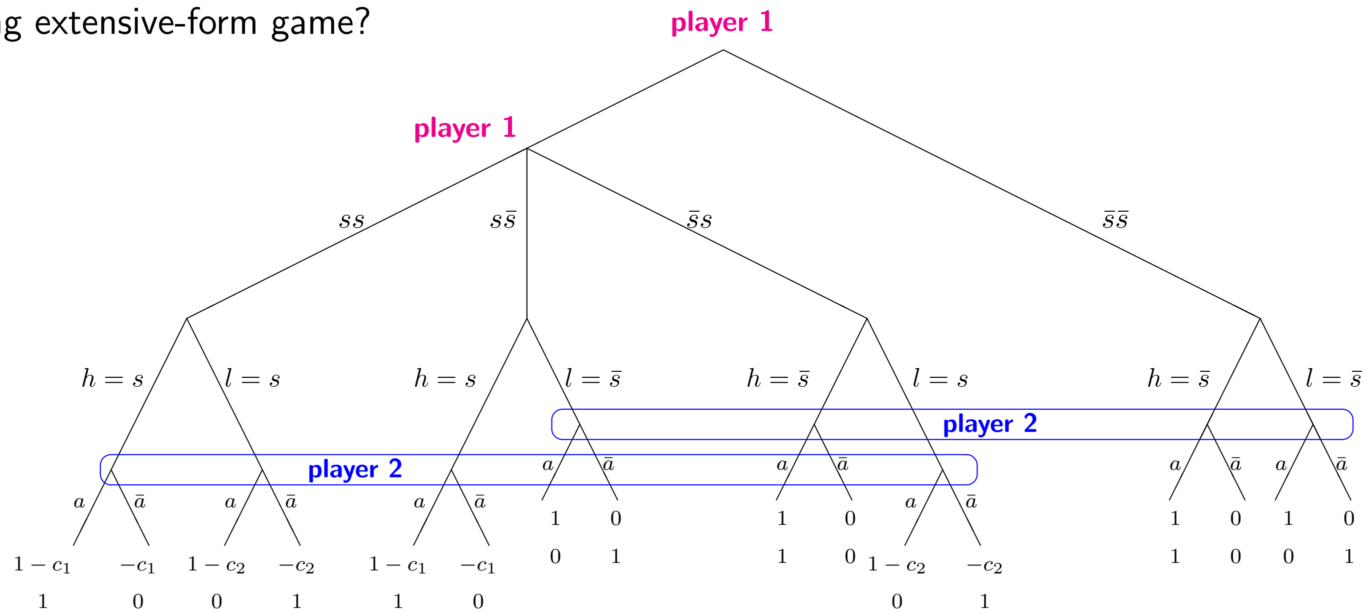

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$\bar{\text{s}}\bar{\text{s}}$	$1,\ p$	$0,\ 1-p$	$1,\ p$	$0,\ 1-p$

Phenomena explained:

When prior is low, $p < 1/2$:

- Partially revealing equilibrium (E1):

  costly signal becomes a means to shape the belief of the other; specifically: "push the belief of the other up" $\longrightarrow$ for of "indirect speech"

- (E1) welfare-improving over "no-signaling" equilibrium outcome (P1).

When prior is high, $p > 1/2$:

- both routinely using the costly signal (P2) and routinely not using costly signal (P3) are strategically and evolutionarily stable equilibrium outcomes
  - overstatement (P2) and understatement (P3)
  - P2: Social tragedy: everybody needs to signal, but signal carries no information!
  - P3 can also be interpreted as "countersignaling"
- co-existence of these two equilibrium outcomes $\rightarrow$ possible source of discrimination: when (P2) or (P3) is linked to some other observable characteristic

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