Problems
on Aumann, Common Knowledge, Dialogues, ...

SOLUTIONS

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1 Introduction

Review

- In probability theory: What is an *event*?

  In probability theory, an event is a subsets of the set of possible states of the world. Events are often denoted by capital letters, such as A, B, C, etc. or A_1, A_2, A_3, etc., and one writes $A \subset \Omega$, etc.

- What is a σ-*algebra*? Give the formal definition.

  Definition: Let Ω be a set. A σ-*algebra* $\mathscr{A}$ on Ω is a family (a set) of subsets of Ω such that:

  (C_1) $\Omega \in \mathscr{A}$.

  (C_2) if $A \in \mathscr{A}$, then $\bar{A} \in \mathscr{A}$

  (C_3^σ) if $(A_n)_{n\in N}$ is a sequence of elements of $\mathscr{A}$ (which by definition are subsets of Ω), then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

- What is the difference between an *algebra* (in French, une *tribu* ou bien une *algb̀re de parties*) and a σ-*algebra* (une σ-*algèbre*)?

  Definition: Let Ω be a set. An *algebra* $\mathscr{A}$ on Ω is a family (a set) of subsets of Ω such that:

  (C_1) $\Omega \in \mathscr{A}$.

  (C_2) if $A \in \mathscr{A}$, then $\bar{A} \in \mathscr{A}$

  (C_3) if $A \in \mathscr{A}$ and $B \in \mathscr{A}$, then $A \cup B \in \mathscr{A}$.

  The difference is in the third condition.

- Show that in the definition of a σ-algebra the condition of closedness with respect to the union [what above we call (C_3^σ)] can be replaced by the condition of closedness with respect to the intersection.

  The condition of closedness with respect to the intersection can be stated as follows:

  (C_4^σ) if $(A_n)_{n\in N}$ is a sequence of elements of $\mathscr{A}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathscr{A}$.

  The two conditions are equivalent, $(C_3^\sigma) \Leftrightarrow (C_4^\sigma)$, if and only if: $(C_3^\sigma) \Rightarrow (C_4^\sigma)$ and $(C_4^\sigma) \Rightarrow (C_3^\sigma)$, that is, if one implies the other.

  Let us show first that $(C_3^\sigma) \Rightarrow (C_4^\sigma)$:

  - Let $(A_n)_{n\in N}$ be a sequence of elements of $\mathscr{A}$ (which by definition is a sequence of subsets of Ω). By C_2, we have $\bar{A}_n \in \mathscr{A}$ for all n.
  - By (C_3^σ), we have $\bigcup_{n=1}^{\infty} \bar{A}_n \in \mathscr{A}$.
  - By Morgan's laws, we have $\overline{\bigcap_{n=1}^{\infty} A_n} \in \mathscr{A}$.
  - By C_2, we have $\overline{\overline{\bigcap_{n=1}^{\infty} A_n}} \in \mathscr{A}$. But $\overline{\overline{\bigcap_{n=1}^{\infty} A_n}} = \bigcap_{n=1}^{\infty} A_n$. Therefore, $\bigcap_{n=1}^{\infty} A_n \in \mathscr{A}$, which is what needed to be shown.

  Let us now first that $(C_4^\sigma) \Rightarrow (C_3^\sigma)$:

  - Let $(A_n)_{n\in N}$ be a sequence of elements of $\mathscr{A}$ (which by definition is a sequence of subsets of Ω). By C_2, we have $\bar{A}_n \in \mathscr{A}$ for all n.
  - By (C_4^σ), we have $\bigcap_{n=1}^{\infty} \bar{A}_n \in \mathscr{A}$.

- By Morgan's laws, we have $\overline{\bigcup_{n=1}^{\infty} \overline{A_n}} \in \mathscr{A}$.
- By C_2, we have $\overline{\overline{\bigcup_{n=1}^{\infty} \overline{A_n}}} \in \mathscr{A}$. But $\overline{\overline{\bigcup_{n=1}^{\infty} \overline{A_n}}} = \bigcup_{n=1}^{\infty} A_n$. Therefore, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$, which is what needed to be shown.

- Show that in the definition of a σ-algebra the condition that Ω itself belong to the collection of subsets of Ω that is to constitute the σ-algebra can be replaced by the condition that the collection of subsets of Ω that is to constitute the σ-algebra be non empty.
- Show that the intersection of two σ-algebras is again a σ-algebra.

2 The formal framework and Aumann's theorem

Problem 1

Let $\Omega = \{a, b, c, d, e, f, g, h, i, j, k\}$ and

$$\mathscr{P}_i \quad = \quad \{\{a, b, g, h\}, \{c, d, i, j\}, \{e, f, k\}\}.$$

Assume that $\omega^\star = c$ is the state of the world that has materialized. Individual i will then *know* that one of the states in $\{c, d, i, j\}$ has happened; in other words, individual i will know that the *event* $\{c, d, i, j\}$ has occurred.

Problem:

(1) Write down all events of which individual i will know that they have occurred when state c has materialized (provided that i has received the information according to the partition $\mathscr{P}_i$).

When state c materializes, i will know that all events that *contain* $\{c, d, i, j\}$, that is, all events $A \subset \Omega$ for which $\{c, d, i, j\} \subset A$, have occurred. For instance: $\{a, c, d, i, j\}$, $\{b, c, d, i, j\}$, $\{a, b, c, d, i, j\}$, etc. In particular i will know that the union of $\{c, d, i, j\}$ with any other information class (cell) of i's partition, that is, the events $\{a, b, c, d, g, h, i, j\}$, $\{c, d, e, f, i, j, k\}$, and $\{a, b, c, d, e, f, g, h, i, j, k\}$, have occurred.

(2) Write down all events of which individual i will know that they have *not* occurred when state c has materialized (provided that i has received the information according to the partition $\mathscr{P}_i$).

When state c materializes, i will know that any event disjoint from $\{c, d, i, j\}$, that is, any event $A \subset \Omega$ for which $\{c, d, i, j\} \cap A = \emptyset$, has not occurred. For instance: $\{a\}$, $\{a, b\}$, $\{a, b, d\}$, etc. In particular i will know that any event that corresponds to an information classes in i's partition different from $\{c, d, i, j\}$ as well as any union over i's information classes different from $\{c, d, i, j\}$, that is, $\{a, b, g, h\}$, $\{e, f, k\}$, and $\{a, b, g, h, e, f, k\}$, has not occurred.

(3) Come up with a "story" that illustrates this example. "Tell" that story using the formalism that we have introduced in class.

The states of the world could represent the persons that one possibly meets when walking through the corridors of the department of Mathematics at a university in London (all men):

a comes from Boston (US),

b from New York (US),

c from Bristol (UK),

d from London (UK),

e from Liverpool (UK),

f from Liverpool (UK),

g from New York (US),

h from Chicago (US),

i from Cambridge (UK),

j from Oxford (UK),

k from Manchester (UK).

The individual modeled by the information partition is a new colleague who does not know the members of the department personally yet. He meets someone in the hall. They greet each other. The new colleague, an American, is able to distinguish three types of accents: an American accent, a standard English accent, and an English accent from the North of England. If he just hears one of his new colleagues greet him, all he will know is that he has just met someone with an American accent, that is, someone who belongs to $\{a, b, g, h\}$, or someone with a standard English accent, that is, someone who belongs to $\{c, d, i, j\}$, or someone with an English accent from the North of England, that is someone who belongs to $\{e, f, k\}$.

(4) Assume that Ω is endowed with a uniform prior probability, that is, the prior probability for every state of the world is $1/11$. Let $A = \{g, i\}$ be the invent of interest. Given the information that i has received according to the partition $\mathscr{P}_i$:

- What is his or her posterior belief (the posterior probability) attributed to A?

$$q_i \quad = \quad \frac{P(A \mid P_i(a))}{P_1(c)} = \frac{p(\{g, i, \} \cap \{c, d, i, j\})}{p(\{c, d, i, j\})} = \frac{p(\{i\})}{p(\{c, d, i, j\})} = \frac{\frac{1}{11}}{\frac{4}{11}} = \frac{1}{4}$$

- Which subset of Ω represents the event that i will attribute to A a posterior probability of $1/4$?

  If any of the states in $\{a, b, g, h\}$ had materialized, individual i would also have come up with a posterior probability of $1/4$ attributed to A, for:

$$\frac{p(\{g, i, \} \cap \{a, b, g, h\})}{p(\{a, b, g, h\})} = \frac{p(\{g\})}{p(\{a, b, g, h\})} = \frac{\frac{1}{11}}{\frac{4}{11}} = \frac{1}{4}$$

  However, if any of the states in $\{e, f, k\}$ had materialized, individual i would have come up with a posterior probability of 0 attributed to A (i would know for sure that A did not happen), for:

$$\frac{p(\{g, i, \} \cap \{e, f, k\})}{p(\{e, f, k\})} = \frac{p(\emptyset)}{p(\{e, f, k\})} = \frac{0}{\frac{4}{11}} = 0$$

  Therefore, the subset of Ω that stands for the event that i will attribute to A a posterior probability of $1/4$ is $\{a, b, g, h\} \cup \{c, d, i, j\} = \{a, b, c, d, g, h, i, j\}$.

  Illustration:

$$\mathscr{P}_i \quad = \quad \{\overbrace{\{a, b, \mathbf{g}, h\}}^{1/4}, \overbrace{\{c, d, \mathbf{i}, j\}}^{1/4}, \overbrace{\{e, f, k\}}^{0}\}.$$

  (States belonging to event A appear in boldface.)

(5) Same question as (4) but under the assumption that the prior probability distribution on Ω is the following:

$$\begin{aligned} p(a) &= 1/5, \\ p(b) &= 1/10, \\ p(c) &= 0, \\ p(d) &= 1/10, \\ p(e) &= 1/5, \\ p(f) &= 1/10, \\ p(g) &= 1/10, \\ p(h) &= 0, \\ p(i) &= 1/10, \\ p(j) &= 0, \\ p(k) &= 1/10. \end{aligned}$$

With this new prior probability distribution on Ω:

- The posterior probability that i will attribute to the event A is:

$$q_i = \frac{P(A \mid P_i(a))}{P_1(c)} = \frac{p(\{g,i,\} \cap \{c,d,i,j\})}{p(\{c,d,i,j\})} = \frac{p(\{i\})}{p(\{c,d,i,j\})} = \frac{\frac{1}{10}}{\frac{2}{10}} = \frac{1}{2}$$

- If any of the states in $\{a,b,g,h\}$ had materialized, individual i would have come up with a posterior probability of 1/4 attributed to A, because:

$$\frac{p(\{g,i,\} \cap \{a,b,g,h\})}{p(\{a,b,g,h\})} = \frac{p(\{g\})}{p(\{a,b,g,h\})} = \frac{\frac{1}{10}}{\frac{4}{10}} = \frac{1}{4}$$

  If any of the states in $\{e,f,k\}$ had materialized, individual i would have come up with a posterior probability of 0 attributed to A (i would know for sure that A did not happen), for:

$$\frac{p(\{g,i,\} \cap \{e,f,k\})}{p(\{e,f,k\})} = \frac{p(\emptyset)}{p(\{e,f,k\})} = \frac{0}{\frac{4}{10}} = 0$$

  Therefore, with this new prior probability distribution on Ω, the subset of Ω that stands for the event that i will attribute to A a posterior probability of 1/4 is $\{c,d,i,j\}$.

Problem 2

Let $\Omega = \{a,b,c,d,e,f,g,h,i,j,k,l,m\}$,

$$\begin{aligned} \mathscr{P}_1 &= \{\{a,b,c,d,e,f\},\{g,h,i,j,k\},\{l\},\{m\}\}, \\ \mathscr{P}_2 &= \{\{a,b,g,h\},\{c,d,i,j\},\{e,f,k\},\{l,m\}\}. \end{aligned}$$

Problem:

(1a) Suppose that $\omega^\star = b$ materializes.

- What are the events that are common knowledge between the two individuals?
  When state b materializes, all events that *contain* $\{a,b,c,d,e,f,g,h,i,j,k\}$, that is, all events $A \subset \Omega$ for which $\{a,b,c,d,e,f,g,h,i,j,k\} \subset A$, will be common knowledge between the two individuals (that is, it will be common knowledge between the two that

they have occurred). Specifically, these will be the events: $\{a,b,c,d,e,f,g,h,i,j,k\}$, $\{a,b,c,d,e,f,g,h,i,j,k,l\}$, $\{a,b,c,d,e,f,g,h,i,j,k,m\}$, and $\{a,b,c,d,e,f,g,h,i,j,k,l,m\}$ (certainly, and trivially, it is common knowledge between the two that one of the states that belong to Ω has occurred). Note that: When state b materializes, $\{a,b,c,d,e,f,g,h,i,j,k\}$, that is, the information class of the meet of the two partitions in which lies the true state of the world, will be the smallest event (subset of Ω) of which it will be common knowledge between the two individuals that the state that has materialized belongs to it and therefore cannot lie outside it.

- What are the events of which the two individuals will have common knowledge that they *did not happen*?

  $\{l,m\}$ and all subsets thereof: $\{l,m\}$, $\{l\}$, $\{m\}$, and $\emptyset$ (they have common knowledge that some state has materialized).

- Can you come up with a "story" that would be captured by the formal model presented here.

  We can expand on the interpretation that we have given for the information partition of problem 1, with the states of the world representing the members of a department, only that there are two more states: l and m. Suppose that l stands for a native French speaker who comes from Montreal, and m for a native French speaker who comes from Paris. On the set of states already present in problem 1, individual 2 (of Problem 2) is like the individual that we have considered in problem 1. Let us assume that the two additional states l and m, to him, being an American, are in the same information class: he recognizes only that the person has something French in his voice, when he speaks English, but he cannot tell if that comes from a native accent from Montreal or from Paris.

  Now suppose that individual 1 is another new colleague in this department. She is a woman and a native French speaker. When she hears another native French speaker speaking English, she can recognizes if the person comes from Montreal or from Paris. On the other hand, she cannot distinguish American English accents from British English accents, or within British English accents. However, she has a good sense for understanding subtleties in dress code and she can tell from the way how people dress if they have done their PhD in the US, which is the case of individuals a, b, c, d, e, f and l, or in the UK, which is the case of individuals g, h, i, j, k, and m. (Note that for l and m this information is potentially useful but not required for her for identifying them because she can perfectly identify l and m on their accents alone.)

  Imagine that these two new colleagues share an office. They confess to each other that so far they do not recognize the members of the department, but they also let the other know what are their "tricks" to get at least some partial information about who is the person when they meet someone at random in the hall (using the information that people carry in their accents, respectively in their dress codes). Now the two new colleagues walk together through the hall and that they encounter b (a person from New York who has done his PhD in New York). Without the two new colleagues saying anything to each other, they will have common knowledge that the person encountered cannot be a native French speaker, but must be someone who belongs to the group of native English speakers.

(1b) Same problem as (1a) but under the assumption that state $\omega^\star = m$ has materialized.

(2) For the two partitions above:

- For each partition, $\mathscr{P}_1$ and $\mathscr{P}_2$, write down the σ-algebra spanned by that partition.

The σ-algebra spanned by $\mathscr{P}_1 = \{\{a,b,c,d,e,f\},\{g,h,i,j,k\},\{l\},\{m\}\}$ is:

$$\begin{aligned}\sigma(\mathscr{P}_1) \quad = \quad & \{\emptyset, \{a,b,c,d,e,f\},\{g,h,i,j,k\},\{l\},\{m\},\\ & \{a,b,c,d,e,f,g,h,i,j,k\},\{a,b,c,d,e,f,l\},\{a,b,c,d,e,f,m\},\\ & \{g,h,i,j,k,l\}\{g,h,i,j,k,m\},\{l,m\}\\ & \{a,b,c,d,e,f,g,h,i,j,k,l\},\{a,b,c,d,e,f,g,h,i,j,k,m\},\{g,h,i,j,k,l,m\},\\ & \{a,b,c,d,e,f,l,m\},\underbrace{\{a,b,c,d,e,f,g,h,i,j,k,l,m\}}_{\Omega}\}\end{aligned}$$

The σ-algebra spanned by $\mathscr{P}_2 = \{\{a,b,g,h\}\{c,d,i,j\},\{e,f,k\},\{l,m\}\}$ is:

$$\begin{aligned}\sigma(\mathscr{P}_2) \quad = \quad & \{\emptyset, \{a,b,g,h\},\{c,d,i,j\},\{e,f,k\},\{l,m\},\\ & \{a,b,g,h,c,d,i,j\},\{a,b,g,h,e,f,k\},\{a,b,g,h,l,m\}\\ & \{c,d,i,j,e,f,k\},\{c,d,i,j,l,m\},\{e,f,k,l,m\},\\ & \{a,b,g,h,c,d,i,j,e,f,k\},\{a,b,g,h,c,d,i,j,l,m\},\{c,d,i,j,e,f,k,l,m\},\\ & \{a,b,g,h,e,f,k,l,m\},\underbrace{\{a,b,g,h,c,d,i,j,e,f,k,l,m\}}_{\Omega}\}\end{aligned}$$

- Which partition is the finest common coarsening (the so-called *meet*) of the two partitions?

  $\hat{\mathscr{P}} = \mathscr{P}_1 \wedge \mathscr{P}_2 = \{\{a,b,c,d,e,f,g,h,i,j,k\},\{l,m\}\}$

- What is the intersection of the two σ-algebras spanned by the two partitions?

  $\sigma(\mathscr{P}_1) \cap \sigma(\mathscr{P}_2) = \{\emptyset, \{a,b,c,d,e,f,g,h,i,j,k\},\{l,m\},\underbrace{\{a,b,c,d,e,f,g,h,i,j,k,l,m,\}}_{\Omega}\}$

- Write down the σ-algebra spanned by the meet of the two partitions. Compare your result to that of the previous point. What are your observations?

  $\sigma(\mathscr{P}_1 \wedge \mathscr{P}_2) = \{\emptyset, \{a,b,c,d,e,f,g,h,i,j,k\},\{l,m\},\underbrace{\{a,b,c,d,e,f,g,h,i,j,k,l,m,\}}_{\Omega}\}$

  We see then that:

$$\sigma(\mathscr{P}_1) \cap \sigma(\mathscr{P}_2) = \sigma(\mathscr{P}_1 \wedge \mathscr{P}_2).$$

(3a) Suppose we are interested in the event $A = \{a, e\}$ and that state b materializes.

- For each individual 1 and 2: What is the posterior probability that the individual in question attributes to the event $A = \{a, e\}$? (Can you relate this to the story that you have given at point (1)?)

$$\begin{aligned} q_1 &= \frac{P(A \mid P_i(b))}{P_1(b)} = \frac{p(\{a,e\} \cap \{a,b,c,d,e,f\})}{p(\{a,b,c,d,e,f\})} = \frac{p(\{a,e\})}{p(\{a,b,c,d,e,f\})} = \frac{\frac{2}{13}}{\frac{6}{13}} = \frac{1}{3} \\ q_2 &= \frac{P(A \mid P_i(b))}{P_1(b)} = \frac{p(\{a,e\} \cap \{a,b,g,h\})}{p(\{a,b,g,h\})} = \frac{p(\{a\})}{p(\{a,b,g,h\})} = \frac{\frac{1}{13}}{\frac{4}{13}} = \frac{1}{4} \end{aligned}$$

  In relation to the story given above, A could be the event that the person is a number theorist, which is, say, the field of study of both individual 1 and 2. Now, when individual 1 meets someone who speaks English with a not French accent and is dressed like someone who did his PhD in the US, she will attribute a probability of 1/3 to that person being a number theorist. When individual 2 meets someone who speaks English with an American accent, he will attribute a probability of 1/4 to that person being a number theorist.

– Will the posterior probabilities that each of the individuals attributes to A be common knowledge between the two individuals?

No. It is easy to see that common knowledge of the posterior probabilities breaks already down at the first level of knowing: When b is the true state of the world, individual 1 thinks it possible, for instance, that state c is the true state of the word. But in that case, individual 2 would have received the information that he true state of the world belong to $\{c, d, i, j\}$, and if that were the case, individual 2 would attribute to A a posterior probability of zero, because,

$$\frac{p(\{a,e\} \cap \{c,d,i,j\})}{p(\{c,d,i,j\})} = \frac{p(\{\emptyset\})}{p(\{c,d,i,j\})} = \frac{0}{\frac{4}{13}} = 0.$$

But when b is the true state of the world, individual 1 thinks it also possible that, for instance, e is the true state of the word. But in that case, individual 2 would have received the information that he true state of the world belong to $\{e, f, k\}$, and if this were the case, individual 2 would attribute to A a posterior probability of $1/3$, because,

$$\frac{p(\{a,e\} \cap \{e,f,k\})}{p(\{e,f,k\})} = \frac{p(\{e\})}{p(\{e,f,k\})} = \frac{\frac{1}{13}}{\frac{3}{13}} = \frac{1}{3}.$$

Therefore, individual 1 does not know that individual 2 attributes to A a posterior probability of $1/4$, and as a consequence—because it is not even simple shared knowledge—cannot be common knowledge. Let us note though that what individual 1 still knows something about the posterior that individual 2 attributes to A, namely that it is $1/4$ or 0 or $1/3$.

Similarly, individual 1 will not know the posterior of individual 2: When b is the true state of the world, individual 2 thinks it possible, for instance, that state g is the true state of the word. But in that case, individual 1 would have received the information that he true state of the world belong to $\{g, h, i, j, k\}$. But if this were the case, individual 1 would attribute to A a posterior probability of zero, because,

$$\frac{p(\{a,e\} \cap \{g,h,i,j,k\})}{p(\{g,h,i,j,k\})} = \frac{p(\{\emptyset\})}{p(\{g,h,i,j,k\})} = \frac{0}{\frac{5}{13}} = 0.$$

Individual 2 knows of the posterior of individual 1 only that it is $1/3$ or 0.

Illustration:

$$\mathscr{P}_1 = \{\overbrace{\{a,b,c,d,e,f\}}^{1/3}, \overbrace{\{g,h,i,j,k\}}^{0}, \{l\}, \{m\}\},$$
$$\mathscr{P}_2 = \{\underbrace{\underbrace{\{a,b,g,h\}}_{1/4}, \underbrace{\{c,d,i,j\}}_{0}, \underbrace{\{e,f,k\}}_{1/3}}_{\hat{P}(b)}, \{l,m\}\}.$$

– Imagine now that individual 1 announces to individual 2 his or her posterior with respect to the event A. Will this have any consequences for the posterior that individual 2 will then attribute to the event A? And if they continued in that fashion, if individual 2 then announced her possibly new posterior ...?

Indeed, if individual 1 announces that her posterior is $1/3$, because it is common knowledge between the two individuals that this is 1 posterior only if the true state lies in $\{a, b, c, d, e, f\}$, it will become common knowledge between the two that the true state lies in this set and that hence all other states can be discarded in common knowledge. It is as if Ω has shrunk and has become $\Omega' = \{a, b, c, d, e, f\}$. If individual 2 now calculates his posterior for the event $A = \{a, e\}$ given this reduced set Ω', he will come up with a posterior of $1/2$.

Illustration:

$$\begin{aligned} \mathscr{P}_1 &= \{a, b, c, d, e, f\}, \{g, h, i, j, k\}, \{l\}, \{m\}\}, \\ \mathscr{P}_2 &= \{\underbrace{\underbrace{\{a, b, , g, h\}}_{1/2}, \underbrace{\{c, d, i, j\}}_{0}, \underbrace{\{e, f, k\}}_{1/2}}_{\hat{P}(b)}, \{l, m\}\}. \end{aligned}$$

If individual 2 announces now that her posterior attributed to A is $1/2$, it will become common knowledge between the two that the true state cannot be in the set $\{c, d\}$, because in that case, individual 2 would have announced a posterior of 0. In other words, it will be common knowledge between the two that the true state of the world must belong to $\Omega'' = \{a, b, e, f\}$.

Illustration:

$$\begin{aligned} \mathscr{P}_1 &= \{\overbrace{\{a, b, c, d, e, f\}}^{1/2}, \{g, h, i, j, k\}, \{l\}, \{m\}\}, \\ \mathscr{P}_2 &= \{\underbrace{\underbrace{\{a, b, g, h\}}_{1/2}, \{c, d, i, j\}, \underbrace{\{e, f, k\}}_{1/2}}_{\hat{P}(b)}, \{l, m\}\}. \end{aligned}$$

If individual 1 announces now that her new posterior attributed to A is $1/2$, individual 2 will know already that this is what individual 1 was going to say, because this posterior will already be common knowledge, because it will be the only possible posterior for any information class of individual 1 that is contained in the class of the meet to which belongs the true state of the world, at this moment of the process (that is, with the reduced universe Ω'' at this step). This announcement will therefore now allow them anymore to discard any states in common knowledge. If individual 2 announces now again $1/2$, this will also not allow individual 1 to learn anything that she did not know before. No more states can be discarded in common knowledge. They have reach a situation in which—with the reduced set Ω''—the conditions that characterize Aumann's result are satisfied.

(3b) Same set of questions as in problem (3a) but under the assumption that state m materializes.

- For each individual 1 and 2: What is the posterior probability that the individual in question attributes to the event $A = \{a, e\}$?

$$\begin{aligned} q_1 &= \frac{P(A \mid P_i(m))}{P_1(m)} = \frac{p(\{a, e\} \cap \{m\})}{p(\{m\})} = \frac{p(\emptyset)}{p(\{m\})} = \frac{0}{\frac{1}{13}} = 0 \\ q_2 &= \frac{P(A \mid P_i(m))}{P_1(m)} = \frac{p(\{a, e\} \cap \{l, m\})}{p(\{l, m\})} = \frac{p(\emptyset)}{p(\{l, m\})} = \frac{0}{\frac{2}{13}} = 0 \end{aligned}$$

- Will the posterior probabilities that each of the individuals attributes to A be common knowledge between the two individuals?

  Yes, because of each of the two individuals: for any information class that is contained in the information class of the meet in which lies the true state of the world, $\{m, l\}$, he or she will come up with the same posterior attributed to A, namely 0. To see this detail: For individual 1, the only other information class that is contained in $\{m, l\}$ is $\{l\}$, and for this class we also have:

$$\frac{p(\{a, e\} \cap \{l\})}{p(\{l\})} = \frac{p(\emptyset)}{p(\{l\})} = \frac{0}{\frac{1}{13}} = 0$$

For individual 2 the argument is even simpler because this individual's information class in which lies the true state of the world coincides with that of the meet in which lies the true state of the world: $\{m, l\}$.

Illustration:

$$\begin{aligned}\mathscr{P}_1 &= \{\{a,b,c,d,e,f\},\{g,h,i,j,k\},\overbrace{\{l\}}^{0},\overbrace{\{m\}}^{0}\},\\ \mathscr{P}_2 &= \{\{a,b,g,h\},\{c,d,i,j\},\{e,f,k\},\underbrace{\{l,m\}}_{0}\}.\end{aligned}$$

– Imagine now that individual 1 announces to individual 2 his or her posterior with respect to the event A. Will this have any consequences for the posterior that individual 2 will then attribute to the event A? And if they continued in that fashion, if individual 2 then announced her possibly new posterior ...?

This will not change anything, because the posteriors are already common knowledge. In fact, in this example, the Aumman conditions are satisfied.

(4a) Same problem as (3a) but with $A = \{m, l\}$ and $\omega^\star = b$.

An illustration might suffice:

$$\begin{aligned}\mathscr{P}_1 &= \{\overbrace{\{a,b,c,d,e,f\}}^{0},\overbrace{\{g,h,i,j,k\}}^{0},\{l\},\{m\}\},\\ \mathscr{P}_2 &= \{\underbrace{\{a,b,g,h\}}_{0},\underbrace{\{c,d,i,j\}}_{0},\underbrace{\{e,f,k\}}_{0},\{l,m\}\}.\end{aligned}$$

One easily sees that the Aumann conditions are fulfilled.

(4b) Same problem as (3b) but with $A = \{m, l\}$ and $\omega^\star = m$.

An illustration might suffice again:

$$\begin{aligned}\mathscr{P}_1 &= \{\{a,b,c,d,e,f\},\{g,h,i,j,k\},\overbrace{\{l\}}^{1},\overbrace{\{m\}}^{1}\},\\ \mathscr{P}_2 &= \{\{a,b,g,h\},\{c,d,i,j\},\{e,f,k\},\underbrace{\{l,m\}}_{1}\}.\end{aligned}$$

Again, one easily sees that the Aumann conditions are fulfilled.

Problem 3

(1) Give the formal definition of a random variable. (Remark: We look for the general definition of a *random variable* and not for the more specific definition of a *real random variable*.)

Definition: Let $(\Omega, \mathscr{A}, P)$ be a probability space and $(E, \mathscr{B})$ a set together with a σ-algebra on it. A random variable is an application $X : \Omega \to E$ such that:

$$X^{-1}(B) \in \mathscr{A}, \text{ for all } B \in \mathscr{B},$$

where $X^{-1} : \mathscr{B} \to \mathscr{P}(\Omega)$ such that $X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\}$, the inverse image of B under X. If the condition is satisfied, then X is said to be measurable with respect to $\mathscr{A}$.

(2) Express the modeling of information about the state of the word by an information partition (as proposed by Aumann 1976) more formally by the concept of a random variable and illustrate that for the information partition given in Problem 1.

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $\mathscr{P}_i$ a partition of Ω modeling the information of individual i. More formally this is to say that i's information is given by the following random variable: $X : \Omega \to \mathscr{P}_i$ such that $X(\omega) = P_i(\omega)$.

For the information partition in problem 1:

$$\begin{aligned}
X(a) &= \{a, b, g, h\} \\
X(b) &= \{a, b, g, h\} \\
X(c) &= \{c, d, i, j\} \\
X(d) &= \{c, d, i, j\} \\
X(e) &= \{e, f, k\} \\
X(f) &= \{e, f, k\} \\
X(g) &= \{a, b, g, h\} \\
X(h) &= \{a, b, g, h\} \\
X(i) &= \{c, d, i, j\} \\
X(j) &= \{c, d, i, j\} \\
X(k) &= \{e, f, k\}
\end{aligned}$$

Problem 4

Suppose the set of possible states of the world Ω is given by the possible outcomes of the throw of a dice, with $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Problem:

(1) Write down the information partition of an individual i who can only distinguish between odd and even outcomes, and of an individual j who can only distinguish between the event that 6 comes up and all the other possible outcomes.

$$\begin{aligned}
\mathscr{P}_i &= \{\{1, 2, 3, 4, 5\}, \{6\}\} \\
\mathscr{P}_j &= \{\{1, 3, 5\}, \{2, 4, 6\}\}.
\end{aligned}$$

(2) Suppose that a six comes up. Which events are common knowledge between the two individuals?

The only event common knowledge between the two will be $\{\{1, 2, 3, 4, 5, 6\} = \Omega$: that is, the trivial event that one of the possible states of the world has materialized.

(3) What is the meet of the two partitions?

$$\hat{\mathscr{P}} = \{\{1, 2, 3, 4, 5, 6\}\}$$

Problem 5

Read Aumann's (1976) "Agreeing to disagree."

Let $\Omega = \{a, b, c, d, e, f\}$, $p(\omega) = 1/6$ for all states of the world, and

$$\begin{aligned} \mathscr{P}_1 &= \{\{a,b\},\{c,d\},\{e,f\}\}, \\ \mathscr{P}_2 &= \{\{a,c\},\{b,d\}\{e\}\{f\}\}. \end{aligned}$$

Assume that the event of interest is $A = \{a\}$ and the state of the word that materializes $\omega^\star = a$.

Problem:

(1) For each individual, 1 and 2, calculate the posterior probability that he or she attributes to the event $A = \{a\}$ given the information that he or she has received through his or her information partition?

(2) Will these posterior beliefs be common knowledge between the two individuals?

(3) What is the meet of the two partitions?

(4) Are the Aumann conditions fulfilled in this example?

Problem 6

Read chapter 2, "Le cadre formel et le théorème d'Aumann," in the lecture notes *Dialogues Aumanniens* (available on the website dedicated to this course).

Problem:

(1) Invent your own example in which the Aumann conditions are satisfied.

(2) Invent your own example in which the Aumann conditions are not satisfied.

3 Direct communication

Problem 7

Read chapter 3, "La communication directe," in the lecture notes *Dialogues Aumanniens* (available on the website dedicated to this course).

Problem:

(1) Write down (in English) the definition of the coarsest common refinement (the *join*) of two partitions.

(2) For the two partitions (i) in Problem 2, (ii) in Problem 4, and (iii) Problem 5 : What is the join of the two partitions?

Problem 8

For any two finite partitions, it is possible to write them in a matrix such that the elements of that matrix are occupied by the classes of the join of the two partitions (with possibly some elements of that matrix unoccupied) and the rows of the matrix correspond to the classes of one partition and the columns to the classes of the other partition.

Problem: Write down this matrix for the two partitions of (i) Problem 2, (ii) Problem 4, and (iii) Problem 5.

4 Indirect communication through the updated beliefs: a Bayesian dialogue

Problem 9

Read Geanakoplos and Polemarchakis's (1982), "We can't disagree forever."

Problem: Invent your own example in which you demonstrate the process of indirect communication as defined by Geanakoplos and Polemarchakis (1982).

(1) In your example investigate in particular if the outcome of the process depends on the order of communication (that is, if it depends on who of the two individuals starts the process of indirect communication by announcing his or her posterior).

(2) Represent the process of elimination of states in Ω that takes place in the background of the process in the matrix presentation of the two individuals' information partitions.

5 Indirect communication through the observation of acts

Problem 10

Read Sebenius and Geanakoplos's (1983), "Don't bet on it: contingent agreements with asymmetric information."

Problem: Invent your own example of a dynamic bet scenario as discussed by Sebenius and Geanakoplos (1983) but set in a context in which Ω is finite.

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